A structural analysis of asymptotic mean-square stability for multi-dimensional linear stochastic differential systems

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Abstract

We are concerned with a linear mean-square stability analysis of numerical methods applied to systems of stochastic differential equations (SDEs) and, in particular, consider the θ-Maruyama and the θ-Milstein method in this context. We propose an approach, based on the vectorisation of matrices and the Kronecker product, that allows us to deal efficiently with the matrix expressions arising in this analysis and that provides the explicit structure of the stability matrices in the general case of linear systems of SDEs. For a set of simple test SDE systems, incorporating different noise structures but only a few parameters, we apply the general results and provide visual and numerical comparisons of the stability properties of the two methods.

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1. Introduction

Stochastic modelling has steadily gained importance in many different application areas, such as finance, neuroscience or electrical circuit engineering, and typically the model systems consist of a large number of stochastic differential equations driven by a large number of noise sources. Thus the area of development and analysis of numerical methods for stochastic differential equations (SDEs) has attracted increasing interest and, in particular, there is an increased need to study numerical methods for their ability to preserve qualitative features of the stochastic continuous system they are developed to approximate. In deterministic numerical analysis a linear stability analysis is usually the first step of an analysis in this direction. The underlying idea for a linear stability analysis is based on the following line of reasoning: one linearises and centres a nonlinear ordinary differential equation $\mathbf{x}'(t) = f(t, \mathbf{x})$ around an equilibrium, the resulting linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ (A the Jacobian of f evaluated at the equilibrium) is then diagonalised and the system thus decoupled, justifying the use of the scalar test equation $x'(t) = \lambda x(t)$, $\lambda \in \mathbb{C}$, for the analysis. We refer to, for example, [15, Ch. IV.2] for more details on this procedure. Starting with a general nonlinear SDE with m noise sources, the same reasoning yields a linear system of SDEs with an m-dimensional driving Wiener process of the form

$$\mathrm{d}X(t) = FX(t)\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\mathrm{d}W_r(t), \quad t > 0\,. \tag{1}$$

Considering the linearised SDE system above when it can be completely decoupled, i.e., all matrices F and G_r are diagonalisable by the same transform and each component of X is only driven by a single Wiener process, provides the basis for treating the scalar linear test equation $\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu_1 X(t)\mathrm{d}W_1(t)$, where its solution is called geometric Brownian motion. Research on stability analysis for SDEs has mainly focused on this test equation and relevant references are given by, e.g., [1, 5, 17, 18, 28]. A first investigation of the case that the linear system is decoupled, but each component of X is driven by m Wiener processes, is presented in [8].
In addition to topics related to stability questions relevant already in deterministic numerical analysis, for example, stiffness, contractivity, dissipativity (we refer to, e.g., [1, 9, 19, 30] for literature concerning stochastic versions of these concepts), matters concerning the stochastic nature of the

equations become important to study and understand as well. For example, these issues involve the different notions of stability in stochastic analysis [6, 17, 18, 28, 33], the consequences of different choices for the implementation of random variables [10], or the effect of a multi-dimensional system or noise source [8, 26, 29].

In this article we aim to contribute to two aspects of mean-square stability theory for stochastic numerics: first we consider the general test system (1) without the assumption of diagonalisability of all matrices, and provide an approach that allows us to manipulate the matrices involved in the stability analysis in an efficient way. Based on the vectorisation of matrices and the Kronecker product we reformulate the stability problems in terms of differential and difference equations of the form $\mathrm{d}\mathsf{E}(Y(t)) = S\,\mathsf{E}(Y(t))\,\mathrm{d}t$ and $\mathsf{E}(Y_{i+1}) = \mathcal{S}\mathsf{E}(Y_i)$ for appropriate vectors $Y(t)$ and Y_i, respectively. As they are the most widely used methods, we examine the θ-Maruyama and the θ-Milstein method in this way in Section 3. Deterministic differential equations describing the mean-square evolution of linear systems of SDEs have already been derived in [2], making use of these as well as of deterministic difference equations in the context of linear stability analyses of numerical methods for SDEs have also been considered before and we already mentioned [26, 29]. The main advantage of the approach presented here lies in the fact that it facilitates using the background and results in linear algebra concerning matrices and their Kronecker products. In this article we illustrate this by providing an efficient derivation of the structure of the matrices S and $\mathcal{S}$ from the matrices F, G_r and the given numerical method. However, we anticipate that our framework will be valuable for further analysis of, e.g., stiffness in a stochastic setting. We would also like to point out that MATLAB has a built-in function for manipulating Kronecker products of matrices, thus providing a simple way to implement the derived structures of matrices for given numerical instances of F, G_r and, for example, compute their eigenvalues.

Second we study the effect of different noise structures and consequently different representations of random variables involved in the numerical method. In particular, the θ-Milstein method requires us to take the discretisation of the double Wiener integral $I_{r_1,r_2} = \int_{t_i}^{t_{i+1}} \int_{t_i}^{s_1} \mathrm{d}W_{r_1}(s_2)\,\mathrm{d}W_{r_2}(s_1)$ into account for the stability analysis. In the case of a single noise or commutative noise the double integral can be rewritten as an expression involving only the Wiener increment and this is the case that has been considered in the literature so far [8, 17, 18, 28], where in [8] we have already shown that for the θ-Milstein method the number of noise terms significantly influences the stability behaviour of the method. In the case that strong approximations of solutions of SDEs are required and the SDE has non-commutative noise, the generation of the Wiener increments and double Wiener integrals according to their joint distribution involves providing a simulation algorithm for the double Wiener integrals I_{r_1,r_2}. Several methods to deal with this situation have been proposed in the literature [12, 13, 22, 21, 27, 32], see also [14] for an implementation in MATLAB and simulation studies. When the purpose of the numerical simulations are weak approximations, then both the Wiener increments and multiple Wiener integrals can be replaced by simpler random variables, e.g., two-point distributed random variables, see [21, 25] and for an investigation of stability issues in this case [10]. In our linear stability analysis of the θ-Milstein scheme applied the system of SDEs (1) we incorporate the case of non-commutative noise and use the discretisation of the double Wiener integrals based on the series expansion of the stochastic Lévy areas proposed in [22, 21]. In general the possibly large number of parameters present in (1) prevents deriving explicit stability conditions that allow an insightful comparison between numerical methods. Therefore, following the strategy employed in [6], in Section 4 we propose a set of three test systems with only a few parameters, a simple structure of the drift matrix and the feature of interest encoded in the diffusion matrices. We thus consider a system with a single Wiener process, one with two Wiener processes where the diffusion matrices satisfy a commutativity condition and another system with two Wiener processes where the diffusion matrices are non-commutative. We apply the results of Section 3 and derive explicit mean-square stability conditions for the SDE systems, the θ-Maruyama method and the θ-Milstein method. In Section 4.7 we provide plots of the three-dimensional stability regions and present further numerical experiments. In Section 5 we summarise and discuss the results.

2. Equations and definitions of asymptotic mean-square stability

We consider the following linear d-dimensional system of stochastic differential equations with m multiplicative noise terms

$$\mathrm{d}X(t) = FX(t)\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\mathrm{d}W_r(t), \quad t \geq t_0 \geq 0, \quad X(t_0) = X_0 \,. \qquad (2)$$

Here, the drift and diffusion matrices are given by $F \in \mathbb{R}^{d\times d}$ and $G_1, \ldots, G_m \in \mathbb{R}^{d\times d}$, respectively, and $W = (W_1, \ldots, W_m)^T$ is an m-dimensional Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq t_0}, \mathbb{P})$. In general it is sufficient to have real-valued entries of the drift and diffusion matrices, as in applications the systems usually are real-valued. In the following, E represents expectation with respect to $\mathbb{P}$ and $\|\cdot\|_{L_2}$ denotes the mean-square norm of a square integrable random variable, i.e., $\|X(t)\|_{L_2} = (\mathsf{E}|X(t)|^2)^{1/2}$. The initial value X_0 is an $\mathcal{F}(t_0)$-measurable random variable, independent of W and with finite second moments, i.e., $\|X(t_0)\|_{L_2} < \infty$. In particular, $X(t_0)$ may be deterministic, then the mean-square norm reduces to the Euclidean vector norm $|X_0|$. Since Eq. (2) is a linear, autonomous stochastic differential equation with constant coefficients, the drift and diffusion satisfy the assumptions of, e.g., [24, Sec. 2.3, Thm. 3.1], and thus a unique strong global solution exists for every initial value. We denote this solution by $X = \{X(t; t_0, X_0),\ t \geq t_0\}$ when we want to emphasise its dependence on the initial data. Moreover, Eq. (2) has the *zero solution* $X(t; t_0, 0) \equiv 0$ as its *equilibrium solution.*

Definition 2.1. *[20, 24] The zero solution of Equation* (2) *is said to be*

1. mean-square stable*, if for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that*

$$\|X(t; t_0, X_0)\|^2_{L_2} < \epsilon\,, \quad t \geq t_0\,,$$

whenever $\|X_0\|^2_{L_2} < \delta$;

2. asymptotically mean-square stable*, if it is mean-square stable and, when $\|X_0\|^2_{L_2} < \delta$,*

$$\|X(t; t_0, X_0)\|^2_{L_2} \to 0 \quad \textit{for} \quad t \to \infty\,.$$

The zero solution is called unstable if it is not stable in the sense above. Definition 2.1 is slightly more general than necessary in the present context, as for the simple linear equation given by Eq. (2) we can take δ arbitrarily large.

In this article we consider two particular one-step approximation methods, the θ-Maruyama and the θ-Milstein method. However, in Section 5 we provide two examples demonstrating that other one-step Maruyama-type or Milstein-type methods can easily be rewritten in the framework presented here. For further background on stochastic numerics we refer to [21, 25]. We denote by X_i, where $X_i = (X_{1,i}, X_{2,i}, \ldots, X_{d,i})^T$, approximations of the solution $X(t_i)$ at discrete time-points on an equidistant grid given by $t_i = t_0 + ih$, $i = 0, 1, 2, \ldots$, where h is the step-size of the method.

The θ-Maruyama scheme for computing approximations X_i of the solution of Eq. (2) takes the form

$$X_{i+1} = X_i + h\,(\theta F X_{i+1} + (1-\theta) F X_i) + \sqrt{h}\sum_{r=1}^{m} G_r\, X_i\, \xi_{r,i}\,,\ i = 0, 1, \dots \quad (3)$$

where we have replaced the Wiener increments $I_r^{t_i,t_{i+1}} = W_r(t_i+h) - W_r(t_i)$ by the scaled random variables $\sqrt{h}\,\xi_{r,i}$. For each r, $r = 1, \dots, m$, the sequence $\{\xi_{r,i}\}_{i\in\mathbb{N}}$ represents one of m independent sequences of mutually independent standard Gaussian random variables, i.e., each $\xi_{r,i}$ is $\mathcal{N}(0,1)$-distributed.

The θ-Milstein method when applied to the linear system (2) reads

$$\begin{aligned} X_{i+1} &= X_i + h\,(\theta F X_{i+1} + (1-\theta) F X_i) + \sqrt{h}\sum_{r=1}^{m} G_r\, X_i\, \xi_{r,i} \\ &\quad + \sum_{r_1,r_2=1}^{m} G_{r_1} G_{r_2}\, X_i\, I_{r_1,r_2}^{t_i,t_{i+1}}\,, \qquad i = 0, 1, \dots\,, \end{aligned} \quad (4)$$

where we have not yet discretised the double Wiener integrals $I_{r_1,r_2}^{t_i,t_{i+1}}$. These integrals are defined by

$$I_{r_1,r_2}^{t_i,t_{i+1}} = \int_{t_i}^{t_{i+1}} \int_{t_i}^{s_1} \mathrm{d}W_{r_1}(s_2)\, \mathrm{d}W_{r_2}(s_1) = \int_{t_i}^{t_{i+1}} (W_{r_1}(s) - W_{r_1}(t_i))\, \mathrm{d}W_{r_2}(s)\,,$$

$r_1, r_2 \in \{1, 2, \dots, m\}$. Depending on the structure of the diffusion matrices G_r, it may be necessary to approximate the multiple Wiener integrals in (4) in different ways, we deal with this issue in Sections 3.5 and 3.6.

For any non-zero initial value X_0, the stochastic difference equations (3) and (4) have a unique solution $X_i(t_0, X_0)$ provided that $(\mathrm{Id} - h\,\theta\,F)$ is invertible. Clearly the difference equations have the same equilibrium $X_i(t_0, X_0) \equiv 0$ as their continuous counterpart Eq. (2) and we can formulate the analogous definition of their mean-square stability below.

Definition 2.2. *The zero solution of a stochastic difference equation generated by a numerical method such as* (3) *or* (4) *applied to Equation* (2) *is said to be*

1. mean-square stable, *if for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that*

$$\|X_i(t_0, X_0)\|_{L_2}^2 < \epsilon\,, \quad i = 0, 1, \dots\,,$$

whenever $\|X_0\|_{L_2}^2 < \delta$;

2. asymptotically mean-square stable, *if it is mean-square stable and, when* $\|X_0\|_{L_2}^2 < \delta$,

$$\|X_i(t_0, X_0)\|_{L_2}^2 \to 0 \quad \textit{for} \quad i \to \infty\,.$$

3. Analysis of asymptotic mean-square stability for linear stochastic systems

In the subsequent analysis of the stability properties of systems of linear stochastic ordinary differential equations and of stochastic difference equations we will make use of the following terms and results, the latter for ease of reference formulated as a lemma.

(i) The vectorisation vec(A) of an $m \times n$ matrix A transforms the matrix A into an $mn \times 1$ column vector obtained by stacking the columns of the matrix A on top of one another.

(ii) The Kronecker product of an $m \times n$ matrix A and a $p \times q$ matrix B is the $mp \times nq$ matrix defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}. \tag{5}$$

(iii) The spectral abscissa $\alpha(A)$ of a matrix A is defined by $\alpha(A) = \max_i \Re(\lambda_i)$, where $\Re$ is the real part of the real or complex eigenvalues λ_i of the matrix A.

(iv) The spectral radius $\rho(A)$ of a matrix A is defined by $\rho(A) = \max_i |\lambda_i|$, where again λ_i are the real or complex eigenvalues of the matrix A.

Lemma 3.1. *[23, Ch. 2]*

(a) $(A+B) \otimes (C+D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$, *if the matrices* $A+B$ *and* $C+D$ *exist;*
(b) *vec*$(ABC) = (C^T \otimes A)$*vec*(B), *when* A, B *and* C *are three matrices, such that the matrix product* ABC *is defined;*
(c) *A special case of (b) is given by vec*$(AB) = (B^T \otimes \mathrm{Id}_m)$*vec*$(A) = (\mathrm{Id}_q \otimes A)$*vec*$(B)$, *where* A *is an* $m \times n$ *matrix,* B *a* $p \times q$ *matrix, and* Id_s *is the* s*-dimensional identity matrix for any* $s \in \mathbb{N}$.

3.1. Mean-square stability analysis for linear stochastic differential systems

The second moment of the solution of Eq. (2), that is the expectation of the matrix-valued process $P(t) = X(t)X(t)^T$, is given by the unique solution of the matrix ordinary differential equation (ODE) [2, Thm. 8.5.1]

$$\mathrm{d}\mathsf{E}(P(t)) = \Big(F\mathsf{E}(P(t)) + \mathsf{E}(P(t))F^T + \sum_{r=1}^{m} G_r \mathsf{E}(P(t))\, G_r^T\Big)\, \mathrm{d}t\,, \tag{6}$$

for $t \geq t_0 \geq 0$, and with initial value $P(t_0) = X_0 X_0^T$. The zero solution of Eq. (2) is asymptotically mean-square stable if and only if the zero solution of the deterministic ordinary differential equation of the second moments (6) is asymptotically stable, see [2, Remark 8.5.5, Section 11] and [20, Remark VI.2.2].

The vectorisation of P yields a d^2-dimensional process $\{Y(t)\}_{t\geq t_0}$ of the form

$$\begin{aligned}\mathrm{vec}(P(t)) &= Y(t) = (Y_1(t), Y_2(t), \ldots, Y_{d^2}(t))^T \qquad (7)\\ &= (X_1^2(t), X_2(t)X_1(t), \ldots, X_d(t)X_1(t),\\ &\quad X_1(t)X_2(t), X_2^2(t), X_3(t)X_2(t), \ldots, X_d(t)X_2(t), \ldots, X_d^2(t))^T.\end{aligned}$$

Applying the vectorisation operation on both sides of (6), and employing Lemma 3.1, parts (b) and (c), we obtain the deterministic linear system of ODEs for the d^2-dimensional vector $\mathsf{E}(Y(t))$

$$\mathrm{d}\mathsf{E}(Y(t)) = S\,\mathsf{E}(Y(t))\,\mathrm{d}t\,, \tag{8}$$

where S is given by

$$S = \mathrm{Id}_d \otimes F + F \otimes \mathrm{Id}_d + \sum_{r=1}^{m} G_r \otimes G_r\,. \tag{9}$$

We refer to S as the stability matrix of the linear ODE system (8) and, in consequence, as the mean-square stability matrix of the linear SDE system (2).

Remark 3.2. *For an $n \times n$ matrix A, the half-vectorisation vech(A) is the $n(n+1)/2 \times 1$ column vector obtained from vec(A) by keeping only the lower triangular part of A. For a symmetric matrix A, vech(A) contains all the distinct elements of A, see [23, Ch. 3]. In [2, Section 11]* *as well as [26, 29] the authors effectively use the half-vectorisation to derive from* (6) *a system of ODEs similar to* (8)*, but for a vector* $\tilde{Y}(t)$ *of the form, e.g. for* $d = 2$, $\tilde{Y}(t) = (X_1^2(t), X_2(t)X_1(t), X_2^2(t))^T$ *instead of* $Y(t) = (X_1^2(t), X_2(t)X_1(t), X_1(t)X_2(t), X_2^2(t))^T$*. Applying the half-vectorisation yields smaller systems to treat in the stability analysis, however, it is then not easy to obtain a structure of the stability matrix such as* (9)*.*

The following is a classical result in the ODE setting, see, e.g., [16, Thm. 9.3].

Lemma 3.3. *The zero solution of the deterministic ODE system* (8) *is asymptotically stable if and only if*

$$\alpha(S) < 0\,. \tag{10}$$

This immediately implies the corresponding result w.r.t. the asymptotic mean-square stability of the zero solution of the stochastic system (2).

3.2. Mean-square stability analysis for systems of linear stochastic difference equations

We now turn to the systems of equations (3) and (4) obtained by applying the θ-Maruyama or the θ-Milstein method to Equation (2). We rewrite each of these stochastic difference equations into an explicit one-step recurrence equation involving a sequence $\{\mathfrak{A}_i\}_{i\geq 0}$ of independent random matrices

$$X_{i+1} = \mathfrak{A}_i X_i, \qquad i = 0, 1, \dots\,. \tag{11}$$

The entries of the matrix $\mathfrak{A}_i$ depend on entries of the drift and diffusion matrices of (2) as well as on the parameter θ of the method and the applied step-size h. For each $i \in \mathbb{N}_0$ the matrix $\mathfrak{A}_i$ also depends on the random variables $\xi_{r,i}$ for all $r = 1, \dots, m$ and possibly further random variables, such as in the case of (4) applied to an SDE with non-commutative noise, see Section 3.6.

To obtain the second moments of the discrete approximation process $\{X_i\}_{i\in\mathbb{N}_0}$ we multiply each side of (11) by $X_{i+1}^T(t)$ and $(\mathfrak{A}_i X_i)^T$, respectively, and take expectations. Then applying the vectorisation operation to both sides and employing Lemma 3.1, part (b), yields

$$\mathsf{E}(Y_{i+1}) = \mathsf{E}(\mathfrak{A}_i \otimes \mathfrak{A}_i)\mathsf{E}(Y_i)\,, \qquad i \in \mathbb{N}_0\,, \tag{12}$$

where the d^2-dimensional discrete process $\{Y_i\}_{i\in\mathbb{N}_0}$ is given by $Y_i = \mathrm{vec}(X_i X_i^T)$.

We set

$$\mathcal{S} = \mathsf{E}(\mathfrak{A} \otimes \mathfrak{A}) \tag{13}$$

and refer to $\mathcal{S}$ as the mean-square stability matrix of the numerical method. The following result is originally due to Bellmann [4], but can also be found in [3, 20].

Lemma 3.4. *The zero solution of the system of linear difference equations* (11) *is asymptotically stable in mean-square if and only if*

$$\rho(\mathcal{S}) < 1\,.$$

Remark 3.5. *We have omitted the index of the matrices $\mathfrak{A}$ in* (13)*, as the sequence $\{\mathfrak{A}_i\}_{i\in\mathbb{N}_0}$ is i.i.d. This notation is consistent with the one used in [4].*

Remark 3.6. *The authors of [26, 29] performed a linear mean-square stability analysis for several numerical methods for test systems of SDEs using the logarithmic matrix norm $\mu_\infty[A]$ and the matrix norm $\|A\|_\infty$ to treat stability matrices such as S and $\mathcal{S}$. The logarithmic matrix norm $\mu_p[A]$ w.r.t. the matrix norm $\|\cdot\|_p$ is defined by*

$$\mu_p[A] = \lim_{\delta\to 0+} (\|\mathrm{Id} + \delta A\|_p - 1)/\delta\,,$$

where Id *is the identity matrix and $\delta \in \mathbb{R}$. For the most common cases $p = 1, 2, \infty$, the corresponding logarithmic matrix norms of a matrix A can be calculated by*

$$\begin{aligned} \mu_1[A] &= \max_j \{a_{jj} + \sum_{i\neq j} |a_{ij}|\}\,, \\ \mu_2[A] &= \frac{1}{2} \max_i \{\lambda_i | \lambda_i \textit{ is eigenvalue of } (A + A^T)\}\,, \\ \mu_\infty[A] &= \max_i \{a_{ii} + \sum_{j\neq i} |a_{ij}|\}\,. \end{aligned}$$

The relation to the spectral abscissa and the spectral radius are provided by the inequalities $\alpha(A) \leq \mu_p[A]$ and $\rho(A) \leq \|A\|_p$ for any induced matrix norm $\|\cdot\|_p$, see [11, 31]. Thus, Lemmas 3.3 and 3.4 provide mean-square stability conditions that are sufficient and necessary, whereas stability conditions in terms of $\mu_\infty[A]$ and $\|A\|_\infty$ yield only sufficient conditions, an illustrative example of this issue is given in [29].

In Sections 3.3 to 3.6 we derive the explicit structure of the stability matrix $\mathcal{S}$ for the methods (3) and (4) applied to the general SDE (2). For the θ-Milstein method we consider three cases of structures of the noise term in (2), i.e., a single driving Wiener process, commutative noise and non-commutative noise. This allows us to study the consequences of the resulting three different representations of the double Wiener integrals $I_{r_1,r_2}^{t_i,t_{i+1}}$ in (4).

3.3. The stability matrix of the θ-Maruyama method

The application of the θ-Maruyama method to the system (2), that is the difference equation (3), formulated as the one-step recurrence (11) reads

$$X_{i+1} = \mathfrak{A}_i X_i \qquad \text{with} \qquad \mathfrak{A}_i = A + \sum_{r=1}^{m} B_r\, \xi_{r,i} \qquad \text{for } i \in \mathbb{N}_0\,. \tag{14}$$

The matrices A and B_r ($r = 1, \ldots, m$) are deterministic and are given by

$$A = (\mathrm{Id} - h\theta F)^{-1}(\mathrm{Id} + h(1-\theta)F) \qquad \text{and} \qquad B_r = (\mathrm{Id} - h\theta F)^{-1}\sqrt{h}G_r\,. \tag{15}$$

Theorem 3.7. *The mean-square stability matrix $\mathcal{S}$ of the θ-Maruyama method applied to* (2) *reads*

$$\mathcal{S} = (A \otimes A) + \sum_{r=1}^{m}(B_r \otimes B_r)\,. \tag{16}$$

Proof. For $\mathfrak{A}_i$ given in (14) and applying Lemma 3.1, part (a), and using $\mathsf{E}(\xi_{r,i}) = 0$ and $\mathsf{E}(\xi_{r,i}^2) = 1$ in the last step, we find that

$$\begin{aligned}
\mathcal{S} &= \mathsf{E}(\mathfrak{A}_i \otimes \mathfrak{A}_i) = \mathsf{E}\left((A + \sum_{r=1}^{m} B_r\, \xi_{r,i}) \otimes (A + \sum_{r=1}^{m} B_r\, \xi_{r,i})\right) \\
&= \mathsf{E}\left((A \otimes A) + (A \otimes \sum_{r=1}^{m} B_r\, \xi_{r,i}) + (\sum_{r=1}^{m} B_r\, \xi_{r,i} \otimes A) + (\sum_{r=1}^{m} B_r\, \xi_{r,i} \otimes \sum_{r=1}^{m} B_r\, \xi_{r,i})\right) \\
&= (A \otimes A) + \sum_{r=1}^{m}(B_r \otimes B_r)\,.
\end{aligned}$$

□

3.4. The stability matrix of the θ-Milstein method for SDEs with a single noise

The θ-Milstein method (4) applied to an SDE (2) with $m = 1$ reads

$$X_{i+1} = X_i + h\,[\theta F X_{i+1} + (1-\theta)FX_i] + \sqrt{h}\,G_1 X_i\,\xi_{1,i} + \frac{1}{2}G_1^2 X_i\,h\,(\xi_{1,i}^2 - 1)\,. \tag{17}$$

As usual in this case, the double Wiener integrals in the last term in (4) have been replaced with terms only involving the Wiener increment by using the identity $I_{1,1}^{t_i,t_i+h} = \frac{1}{2}((I_1^{t_i,t_i+h})^2 - h)$. This case corresponds to the setting for the stability analysis of scalar equations found in the literature so far, see [17, 28].

Hence, the explicit recurrence (11) takes the form

$$X_{i+1} = \mathfrak{A}_i X_i \qquad \text{with } \mathfrak{A}_i = \bar{A} + B\xi_i + C\xi_{1,i}^2\,, \tag{18}$$

where

$$\begin{aligned} \bar{A} &= (\mathrm{Id} - h\theta F)^{-1}\Big(\mathrm{Id} + h(1-\theta)F - \frac{1}{2}hG_1^2\Big) \;=\; A - C\,,\\ B &= B_1 = (\mathrm{Id} - h\theta F)^{-1}\Big(\sqrt{h}G_1\Big)\,,\\ C &= (\mathrm{Id} - h\theta F)^{-1}\Big(\frac{1}{2}hG_1^2\Big)\,. \end{aligned}$$

The matrices A and B_1 are the same as those for the θ-Maruyama method given in (15). With this notation we can state the following theorem.

Theorem 3.8. *For the θ-Milstein method applied to the system* (2) *with $m = 1$ the mean-square stability matrix of the method is given by*

$$\mathcal{S} = (A \otimes A) + (B \otimes B) + 2(C \otimes C)\,. \tag{19}$$

Proof. Using the definitions of the matrices above, part (a) of Lemma 3.1, as well as $\mathsf{E}(\xi_{1,i}) = 0$, $\mathsf{E}(\xi_{1,i}^2) = 1$, $\mathsf{E}(\xi_{1,i}^3) = 0$, and $\mathsf{E}(\xi_{1,i}^4) = 3$, we find that

$$\begin{aligned} \mathcal{S} &= \mathsf{E}(\mathfrak{A}_i \otimes \mathfrak{A}_i) \quad = \quad \mathsf{E}\left((\bar{A} + B\xi_{1,i} + C\xi_{1,i}^2) \otimes (\bar{A} + B\xi_{1,i} + C\xi_{1,i}^2)\right)\\ &= (\bar{A} \otimes \bar{A}) + (\bar{A} \otimes C) + (B \otimes B) + (C \otimes \bar{A}) + 3(C \otimes C)\\ &= (A \otimes A) + (B \otimes B) + 2(C \otimes C)\,. \end{aligned}$$

□

3.5. The stability matrix of the θ-Milstein method for SDEs with commutative noise

The commutativity condition on the diffusion matrices in the case of the linear system (2) takes the form $G_{r_1}G_{r_2} = G_{r_2}G_{r_1}$ for all $r_1, r_2 = 1, 2, \ldots, m$, see for example [21, Sec. 10.3]. Together with the identity

$$I_{r_1,r_2} \quad = \quad I_{r_1}I_{r_2} - I_{r_2,r_1}, \tag{20}$$

for $r_1 \neq r_2$, we can write the θ-Milstein method for a d-dimensional linear system (2) with m noise terms subject to the above commutativity condition as

$$\begin{aligned} X_{i+1} \quad = \quad & X_i + h\left(\theta F X_{i+1} + (1-\theta)FX_i\right) + \sum_{r=1}^{m} \sqrt{h}\, G_r X_i\, \xi_{r,i} \\ & + \sum_{r=1}^{m} \frac{1}{2} h\, G_r^2 X_i(\xi_{r,i}^2 - 1) + \sum_{\substack{r_1,r_2=1 \\ r_1 \neq r_2,\ r_2 > r_1}}^{m} h\, G_{r_1}G_{r_2}X_i\, \xi_{r_1,i}\xi_{r_2,i}\,, \quad i \in \mathbb{N}_0 \end{aligned} \tag{21}$$

To the best of our knowledge the only reference dealing with a stability analysis for SDEs with commutative noise and several noise sources so far is [8]. The recurrence (11) is now given by

$$X_{i+1} = \mathfrak{A}_i\, X_i \quad \text{with} \quad \mathfrak{A}_i = \bar{A} + \sum_{r=1}^{m} B_r\, \xi_{r,i} + \sum_{r_1,r_2=1}^{m} C_{r_1,r_2}\, \xi_{r_1,i}\, \xi_{r_2,i}\,, \tag{22}$$

where $\bar{A}$, B, and C are deterministic matrices determined by

$$\begin{aligned} \bar{A} \quad &= \quad A - (\mathrm{Id} - h\theta F)^{-1}\Big(\sum_{r=1}^{m}\frac{1}{2}h\, G_r^2\Big) = A - \sum_{r=1}^{m} C_{r,r}\,, \\ B_r \quad &= \quad (\mathrm{Id} - h\theta F)^{-1}\Big(\sqrt{h}\, G_r\Big)\,, \\ C_{r_1,r_2} \quad &= \quad (\mathrm{Id} - h\theta F)^{-1}\Big(\frac{1}{2}h\, G_{r_1}G_{r_2}\Big)\,. \end{aligned} \tag{23}$$

Theorem 3.9. *The mean-square stability matrix of the θ-Milstein method applied to the system* (2) *with commutative noise is given by*

$$\mathcal{S} = (A \otimes A) + \sum_{r=1}^{m}(B_r \otimes B_r) + 2\sum_{r=1}^{m}(C_{r,r} \otimes C_{r,r}) + \left(\sum_{\substack{r_1,r_2=1 \\ r_1 \neq r_2}}^{m} C_{r_1,r_2} \otimes \sum_{\substack{r_1,r_2=1 \\ r_1 \neq r_2}}^{m} C_{r_1,r_2}\right). \tag{24}$$

Proof. Analogous computations as in the previous proofs, using the definitions of the matrices above, part (a) of Lemma 3.1, and $\mathsf{E}(\xi_{r,i}) = 0$, $\mathsf{E}(\xi_{r,i}^2) = 1$, $\mathsf{E}(\xi_{r,i}^3) = 0$, and $\mathsf{E}(\xi_{r,i}^4) = 3$, yield the desired result. □

3.6. The stability matrix of the θ-Milstein method for SDEs with non-commutative noise

When the diffusion term in the SDE (2) is non-commutative, i.e., the condition $G_{r_1}G_{r_2} = G_{r_2}G_{r_1}$ for all $r_1, r_2 = 1, 2, \ldots, m$, is not satisfied, then it is not possible, as before, to rearrange the θ-Milstein method (4) into a form that contains only Wiener increments. When the purpose of simulating solutions of an SDE is to obtain strong approximations, that is when the approximation error is measured in the L_2-norm and the θ-Milstein method is expected to perform with mean-square order 1, then it is also not possible to replace the double Wiener integrals by expressions containing 2-point or 3-point distributed random variables as for weak approximations, see, e.g., [21, Ch. 14], [25, Ch. 2.1] or [10]. In this case it is necessary to simultaneously generate the random variables $I_r^{t_i,t_{i+1}}$ and $I_{r_1,r_2}^{t_i,t_{i+1}}$ required in each step of the simulation according to their joint distribution. To simplify the notation, subsequently we make use of the fact that the joint distribution of all these random variables only depends on the step-size h and not on t_i, t_{i+1}, and we thus omit the dependence on t_i, t_{i+1}. Based on a series expansion of the so-called Lévy stochastic area integrals, where these are defined by

$$A_{r_1,r_2} = \frac{1}{2}(I_{r_1,r_2} - I_{r_2,r_1}) \,, \quad r_1, r_2 = 1, \ldots, m \,,$$

the authors in [22] have proposed the following simultaneous representation of the required random variables:

$$\begin{aligned} I_r &\sim \mathcal{N}(0,h) \,, \quad \text{which we rescale again to } \sqrt{h}\,\xi_r \sim \mathcal{N}(0,1) \,, \\ I_{r_1,r_2} &= \frac{1}{2}(I_{r_1}I_{r_2} - h\delta_{r_1r_2}) + A_{r_1,r_2}(h) = \frac{1}{2}\,h\,(\xi_{r_1}\xi_{r_2} - \delta_{r_1r_2}) + A_{r_1,r_2}(h) \,, \\ A_{r_1,r_2}(h) &= \frac{h}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left\{\chi_{r_1,k}(\zeta_{r_2,k} + \sqrt{2}\xi_{r_2}) - \chi_{r_2,k}(\zeta_{r_1,k} + \sqrt{2}\xi_{r_1})\right\} \,. \end{aligned} \qquad (25)$$

Here $\chi_{r,k}$ and $\zeta_{r,k}$ are $\mathcal{N}(0,1)$-distributed independent random variables for all $r = 1, \ldots, m$ and $k = 1, 2, \ldots$, and δ_{ij} is the Kronecker delta. For a practical implementation the infinite sum representing $A_{r_1,r_2}(h)$ has to be truncated after, say, p terms and we denote the resulting finite sum by $A^p_{r_1,r_2}(h)$. Obviously, the choice of the number p of terms kept of the series determines the mean-square accuracy of the approximation of the Lévy stochastic area integrals and in turn the mean-square accuracy of the θ-Milstein scheme. Choosing $p \geq 1/h$ yields the desired strong convergence of order 1 of the θ-Milstein scheme, see, e.g., [21].

In the following we use the representation (25), or rather the truncated series $A^p_{r_1,r_2}(h)$, in the formulation of the θ-Milstein method (4) and proceed with the mean-square stability analysis for the θ-Milstein method. We rewrite (4) into the explicit form

$$\begin{aligned} X_{i+1} &= \mathfrak{A}_i\, X_i \qquad \text{with} \\ \mathfrak{A}_i &= \bar{A} + \sum_{r=1}^{m} B_r\, \xi_{r,i} + \sum_{r_1,r_2=1}^{m} C_{r_1,r_2}\, \xi_{r_1,i}\, \xi_{r_2,i} + \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} D_{r_1,r_2}\, A^p_{r_1,r_2}(h)\,, \end{aligned} \tag{26}$$

where $\bar{A}$, B_r, and C_{r_1,r_2} are deterministic matrices already defined in (23), and D_{r_1,r_2} is a deterministic matrix given by

$$D_{r_1,r_2} \;=\; (\mathrm{Id} - h\theta F)^{-1}\Big(G_{r_1} G_{r_2}\Big) \;=\; \frac{2}{h}\, C_{r_1,r_2}\,.$$

Theorem 3.10. *For the θ-Milstein approximation of the system* (2) *with non-commutative noise the mean-square stability matrix of the method is given by*

$$\begin{aligned} \mathcal{S} \;=\;& (A\otimes A) + \sum_{r=1}^{m}(B_r\otimes B_r) + 2\sum_{r=1}^{m}(C_{r,r}\otimes C_{r,r}) + \Bigg(\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} C_{r_1,r_2} \otimes \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} C_{r_1,r_2}\Bigg) \\ &+ \frac{6}{\pi^2}\Bigg(\Bigg(\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1<r_2}}^{m} C_{r_1,r_2} - \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1>r_2}}^{m} C_{r_1,r_2}\Bigg)\otimes\Bigg(\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1<r_2}}^{m} C_{r_1,r_2} - \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1>r_2}}^{m} C_{r_1,r_2}\Bigg)\Bigg)\sum_{k=1}^{p}\frac{1}{k^2}\,, \end{aligned} \tag{27}$$

where the matrices are defined above and p determines the number of terms retained in the series (25) *for the approximation of the Lévy stochastic area integrals.*

Proof. The derivation of the first four terms in the right-hand side of (27) is identical to those in the proof of Theorem 3.9. Considering the Kronecker products involving the last term in (26), we obtain the identities

$$\mathsf{E}\Big(\bar{A} \otimes \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} D_{r_1,r_2}\, A^p_{r_1,r_2}(h)\Big) = 0\,,$$

$$\mathsf{E}\Big(\sum_{r=1}^{m} B_r\, \xi_{r,i} \otimes \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} D_{r_1,r_2}\, A^p_{r_1,r_2}(h)\Big) = 0\,,$$

$$\mathsf{E}\Big(\sum_{r_1,r_2=1}^{m} C_{r_1,r_2}\, \xi_{r_1,i}\, \xi_{r_2,i} \otimes \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} D_{r_1,r_2}\, A^p_{r_1,r_2}(h)\Big) = 0\,.$$

These follow from the definition of $A^p_{r_1,r_2}(h)$ as then any power of $\xi_{r,i}$ appearing in these expressions is multiplied by exactly one independent $\mathcal{N}(0,1)$-distributed random variable $\chi_{r,k}$ or $\zeta_{r,k}$, yielding expectation zero in the results above. The only non-trivial contribution to (27) is due to the Kronecker product of the last term in (26) with itself. As we have

$$\mathsf{E}\Big(D_{r_1,r_2}\, A^p_{r_1,r_2}(h) \otimes D_{r_1,r_2}\, A^p_{r_1,r_2}(h)\Big) = \frac{4}{h^2}\,(C_{r_1,r_2} \otimes C_{r_1,r_2})\, 6\, \frac{h^2}{(2\pi)^2} \sum_{k=1}^{p} \frac{1}{k^2}\,,$$

$$\mathsf{E}\Big(D_{r_1,r_2}\, A^p_{r_1,r_2}(h) \otimes D_{r_2,r_1}\, A^p_{r_2,r_1}(h)\Big) = \frac{4}{h^2}\,(C_{r_1,r_2} \otimes C_{r_2,r_1})\, (-6)\, \frac{h^2}{(2\pi)^2} \sum_{k=1}^{p} \frac{1}{k^2}\,,$$

where for all other combinations, such as $\mathsf{E}(D_{r_1,r_2}\, A^p_{r_1,r_2}(h) \otimes D_{r_1,r_3}\, A^p_{r_1,r_3}(h))$, the resulting expectation is again zero, we obtain the final contribution, i.e., the last term in (27), as

$$\begin{aligned}
&\mathsf{E}\ \Big(\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} D_{r_1,r_2}\, A^p_{r_1,r_2}(h) \otimes \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} D_{r_1,r_2}\, A^p_{r_1,r_2}(h)\Big) \\
&\quad = \frac{6}{\pi^2}\Big(\Big(\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1<r_2}}^{m} C_{r_1,r_2} - \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1>r_2}}^{m} C_{r_1,r_2}\Big) \otimes \Big(\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1<r_2}}^{m} C_{r_1,r_2} - \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2,\ r_1>r_2}}^{m} C_{r_1,r_2}\Big)\Big) \sum_{k=1}^{p} \frac{1}{k^2}\,.
\end{aligned}$$

□

Remark 3.11. *Although we have in this article only explicitly treated the θ-Maruyama and θ-Milstein method, the results are more general in the sense* *that they can be applied to other Maruyama- and Milstein-type methods when these can be represented as the explicit recursion* (14) *or* (22)*, respectively. The structure of the stability matrix* $\mathcal{S}$ *remains unchanged, only the specific definitions of the matrices* A*,* B_r*,* C_{r_1,r_2} *and* D_{r_1,r_2} *need to be adjusted. We provide two illustrative examples, an explicit 3-stage Runge-Kutta-Maruyama method developed in [7] and the derivative-free Milstein method given in [21], for which the mean-square stability for a scalar linear equation was analysed in [28].*

Example 3.12. *The* 3*-stage explicit Runge-Kutta Maruyama method applied to the linear system* (2) *is given by*

$$X_{i+1} = X_i + h\sum_{s=1}^{3}\beta_s F H_s + \sqrt{h}\sum_{r=1}^{m} G_r X_i \xi_{r,i}\,, \qquad i \geq 0$$

with stages values $H_s = X_i + h\sum_{j=1}^{s-1} a_{ij} F H_j$*, where the parameters* α_{ij} *and* β_i *have to satisfy certain order conditions, see [7]. For the stage values we get iteratively*

$$\begin{aligned}
H_1 &= X_i \\
H_2 &= X_i + a_{21}hFH_1 = (Id + a_{21}hF)\,X_i \\
H_3 &= X_i + a_{31}hFH_1 + a_{32}hFH_2 = (\mathrm{Id} + a_{31}hF + a_{32}hF + a_{32}a_{21}h^2F^2)\,X_i\,,
\end{aligned}$$

and the Runge-Kutta-Maruyama method reads

$$\begin{aligned}
X_{i+1} &= \Big(\mathrm{Id} + hF\sum_{s=1}^{3}\beta_s + h^2F^2(\beta_2 a_{21} + \beta_3 a_{31} + \beta_3 a_{32}) + h^3F^3(\beta_3 a_{32} a_{21})\Big)\,X_i \\
&\quad + \sqrt{h}\sum_{r=1}^{m} G_r X_i \xi_{r,i}\,, \qquad i \geq 0\,.
\end{aligned}$$

Hence the matrices A *and* B_r *for the stability matrix* $\mathcal{S}$ *are given by*

$$\begin{aligned}
A &= \mathrm{Id} + hF\sum_{s=1}^{3}\beta_s + h^2F^2(\beta_2 a_{21} + \beta_3 a_{31} + \beta_3 a_{32}) + h^3F^3(\beta_3 a_{32} a_{21})\,, \\
B_r &= \sqrt{h}\,G_r, \quad r = 1, \ldots, m.
\end{aligned}$$

Example 3.13. *The derivative-free Milstein method applied to the linear system* (2) *reads for* $i \geq 0$

$$X_{i+1} \;=\; X_i + hFX_i + \sqrt{h}\sum_{r=1}^{m} G_r X_i \xi_{r,i} + \frac{1}{\sqrt{h}} \sum_{r_1,r_2=1}^{m} \left(G_{r_2}\Upsilon_{r_1} - G_{r_2}X_i\right) I_{r_1,r_2}^{t_i,t_{i+1}}\,,$$

where the value Υ_r *can be chosen as* $\Upsilon_r = X_i + \sqrt{h}\,G_r X_i$*, see [21]. In the case of commutative noise terms that method reduces to* (21) *and we can apply the same stability calculation as for the classical method* (21)*.*

4. An application of the results to several test systems

In the previous section we have derived the general structure of the stability matrices in the case of arbitrary linear systems and θ-Maruyama and θ-Milstein methods applied to them. In practice, for a given system and a given method, this allows us to determine the corresponding stability matrices S and $\mathcal{S}$ and calculate numerically the spectral abscissa of the stability matrix S as well as the spectral radius of the matrix $\mathcal{S}$, e.g. using MATLAB, possibly for several values of the step-size h, to determine for which step-size the method preserves the stability behaviour of the continuous problem. Theoretically one could also, using a symbolic computing package like MAPLE or MATHEMATICA, obtain these results, that is the spectral abscissa of the stability matrix S and the spectral radius of the matrix $\mathcal{S}$, for arbitrary systems and arbitrary parameters in the method. However, the results are too complex to yield insight into the properties of methods or to compare the performance of methods. Therefore, in this section we will focus on small test systems and employ the results of the last section to investigate the effect that the noise structure has on the stability properties of the two methods.

4.1. Two-dimensional test equations with different noise structures

We consider three test systems all of which have the same, very simple, drift term. The systems differ in the number of noise terms and in the structure of the diffusion matrices, which allows us to investigate the effect that the different discretisations of the diffusion in the θ-Milstein approximations have on the stability properties of the method.

The first test system is a two-dimensional system with a single noise term:

$$\mathrm{d}X(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X(t)\mathrm{d}t + \begin{pmatrix} \sigma & \epsilon \\ \epsilon & \sigma \end{pmatrix} X(t)\mathrm{d}W_1(t)\,; \qquad (28)$$

the second test system is a two-dimensional system with two commutative noise terms:

$$\mathrm{d}X(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X(t)\mathrm{d}t + \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} X(t)\mathrm{d}W_1(t) + \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} X(t)\mathrm{d}W_2(t)\,; \tag{29}$$

and the third one has two non-commutative noise terms:

$$\mathrm{d}X(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X(t)\mathrm{d}t + \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} X(t)\mathrm{d}W_1(t) + \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} X(t)\mathrm{d}W_2(t)\,. \tag{30}$$

We remark that in [29] a system similar to (28) has been used for a mean-square stability analysis of the Euler-Maruyama method, and in [6] the test system (29) has been used to analyse and compare the asymptotic behaviour of the θ-Maruyama scheme with respect to mean-square and almost sure stability.

4.2. Stability conditions for the differential test systems

We first present the results of Section 3.1 applied to the systems (28) – (30). The mean-square stability matrices can easily be computed using the formula (9) and for the test system (28) this yields

$$S_{(28)} = \begin{pmatrix} 2\lambda+\sigma^2 & \sigma\epsilon & \epsilon\sigma & \epsilon^2 \\ \sigma\epsilon & 2\lambda+\sigma^2 & \epsilon^2 & \epsilon\sigma \\ \epsilon\sigma & \epsilon^2 & 2\lambda+\sigma^2 & \sigma\epsilon \\ \epsilon^2 & \epsilon\sigma & \sigma\epsilon & 2\lambda+\sigma^2 \end{pmatrix},$$

and for the test systems (29) and (30) the matrices read

$$S_{(29),(30)} = \begin{pmatrix} 2\lambda+\sigma^2 & 0 & 0 & \epsilon^2 \\ 0 & 2\lambda\pm\sigma^2 & \mp\epsilon^2 & 0 \\ 0 & \mp\epsilon^2 & 2\lambda\pm\sigma^2 & 0 \\ \epsilon^2 & 0 & 0 & 2\lambda+\sigma^2 \end{pmatrix},$$

where in $\pm$, $\mp$ the top signs determine $S_{(29)}$ and the bottom signs determine $S_{(30)}$.

Lemma 4.1.

1. *The zero solution of* (28) *is asymptotically mean-square stable if and only if*

$$\lambda + \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} + |\epsilon\sigma| < 0\,. \tag{31}$$

2. *The zero solutions of* (29) *as well as* (30) *are asymptotically mean-square stable if and only if*

$$\lambda + \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} < 0\,. \tag{32}$$

Proof. In order to apply Lemma 3.3, we have computed with MAPLE the eigenvalues of the stability matrices $S_{(28)}$, $S_{(29)}$ and $S_{(30)}$. For the test system (28) the eigenvalues of the stability matrix $S_{(28)}$ were computed as $2\lambda + \sigma^2 + 2\sigma\epsilon + \epsilon^2$, $2\lambda + \sigma^2 - 2\sigma\epsilon + \epsilon^2$, and $\lambda - \epsilon^2 + \sigma^2$, where the latter one is a double eigenvalue. Condition (31) follows by inspection. For the test systems (29) the stability matrix $S_{(29)}$ has two double eigenvalues $2\lambda + \sigma^2 + \epsilon^2$ and $2\lambda + \sigma^2 - \epsilon^2$. For the test system (30) the eigenvalues of $S_{(30)}$ have been evaluated as $2\lambda + \sigma^2 + \epsilon^2$, $2\lambda + \sigma^2 - \epsilon^2$, $2\lambda - \sigma^2 + \epsilon^2$, and $2\lambda - \sigma^2 - \epsilon^2$. In both cases the largest eigenvalue is given by $2\lambda + \sigma^2 + \epsilon^2$, which gives (32). □

In Sections 4.3 to 4.6 we employ the results of Sections 3.3 to 3.6 to determine the stability matrices of the stochastic difference systems arising from the application of the numerical methods to the test systems (28) to (30). Lemma 3.4 is applied to provide stability conditions in all the cases.

4.3. Stability conditions for the θ-Maruyama method

The stability matrices $\mathcal{S}^{\rm Mar}_{(28)}, \mathcal{S}^{\rm Mar}_{(29)}$ and $\mathcal{S}^{\rm Mar}_{(30)}$, resulting from the application of the θ-Maruyama method to the corresponding test system, are obtained via the formula (16). Thus, in the case of the test system (28) with a single noise term the stability matrix is given by

$$\begin{aligned}
\mathcal{S}^{\rm Mar}_{(28)} &= (A \otimes A) + (B_1 \otimes B_1) \\
&= \frac{1}{(1-h\theta\lambda)^2}\begin{pmatrix} r^2(\lambda h) + \sigma^2 h & \sigma\epsilon h & \epsilon\sigma h & \epsilon^2 h \\ \sigma\epsilon h & r^2(\lambda h) + \sigma^2 h & \epsilon^2 h & \epsilon\sigma h \\ \epsilon\sigma h & \epsilon^2 h & r^2(\lambda h) + \sigma^2 h & \sigma\epsilon h \\ \epsilon^2 h & \epsilon\sigma h & \sigma\epsilon h & r^2(\lambda h) + \sigma^2 h \end{pmatrix},
\end{aligned}$$

for the test system (29) the stability matrix is obtained as

$$\mathcal{S}^{\rm Mar}_{(29)} = (A \otimes A) + (B_1 \otimes B_1) + (B_2 \otimes B_2) = \begin{pmatrix} a + b_1 & 0 & 0 & b_2 \\ 0 & a + b_1 & -b_2 & 0 \\ 0 & -b_2 & a + b_1 & 0 \\ b_2 & 0 & 0 & a + b_1 \end{pmatrix},$$

and for the third test system (30) formula (16) yields

$$\mathcal{S}^{\mathrm{Mar}}_{(30)} = (A \otimes A) + (B_1 \otimes B_1) + (B_2 \otimes B_2) = \begin{pmatrix} a+b_1 & 0 & 0 & b_2 \\ 0 & a-b_1 & b_2 & 0 \\ 0 & b_2 & a-b_1 & 0 \\ b_2 & 0 & 0 & a+b_1 \end{pmatrix}.$$

Here A, B_1 and B_2 are given by (15), the function $r(x)$ denotes $1+(1-\theta)x$ and

$$a = \left(1 + \frac{h\lambda}{1-h\theta\lambda}\right)^2, \quad b_1 = \frac{\sigma^2 h}{(1-h\theta\lambda)^2}, \quad \text{and} \quad b_2 = \frac{\epsilon^2 h}{(1-h\theta\lambda)^2}. \tag{33}$$

We obtain the following result.

Corollary 4.2.

1. *The zero-solution of the stochastic difference equation generated by the θ-Maruyama method applied to the test-equation* (28) *is asymptotically mean-square stable if and only if*

$$\frac{(1+(1-\theta)h\lambda)^2 + h(|\sigma|+|\epsilon|)^2}{(1-\lambda h\theta)^2} < 1. \tag{34}$$

2. *The zero-solution of the stochastic difference equation generated by the θ-Maruyama method applied to the test-equation* (29) *as well as to* (30) *is asymptotically mean-square stable if and only if*

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2+\epsilon^2)}{(1-\lambda h\theta)^2} < 1. \tag{35}$$

Proof. We have used Maple to determine the eigenvalues of the stability matrices and then apply Lemma 3.4. We have found that the stability matrix $\mathcal{S}^{\mathrm{Mar}}_{(28)}$ has three distinct eigenvalues, given by

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2-\epsilon^2)}{(1-\lambda h\theta)^2} \quad \text{and} \quad \frac{(1+(1-\theta)h\lambda)^2 + h(\sigma\pm\epsilon)^2}{(1-\lambda h\theta)^2},$$

and the maximum of the two latter ones yields (34). The stability matrix $\mathcal{S}^{\mathrm{Mar}}_{(29)}$ has two distinct eigenvalues, each with multiplicity two, given by

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2\pm\epsilon^2)}{(1-\lambda h\theta)^2},$$

and the stability matrix $\mathcal{S}^{\text{Mar}}_{(30)}$ has four distinct eigenvalues, given by

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\pm\sigma^2 \pm \epsilon^2)}{(1-\lambda h\theta)^2}\,.$$

In both cases upon inspection the spectral radius of the stability matrices is given by the left hand-side of (35). □

4.4. The stability condition for the θ-Milstein method for the test system with a single noise

For the test system (28) with a single noise term, the stability matrix is given by formula (19) as $\mathcal{S}^{\text{Mil}}_{(28)} = (A\otimes A) + (B_1 \otimes B_1) + 2(C\otimes C)$, where the first two terms are identical to $\mathcal{S}^{\text{Mar}}_{(28)}$ in the previous section. With

$$C = \frac{\frac{1}{2}h}{1-h\theta\lambda}\begin{pmatrix} \sigma^2+\epsilon^2 & 2\sigma\epsilon \\ 2\sigma\epsilon & \sigma^2+\epsilon^2 \end{pmatrix},$$

the remaining part reads

$$2(C\otimes C) \;=\; \frac{2\cdot\frac{1}{4}h^2}{(1-h\theta\lambda)^2}\begin{pmatrix} (\sigma^2+\epsilon^2)^2 & 2\sigma\epsilon(\sigma^2+\epsilon^2) & 2\sigma\epsilon(\sigma^2+\epsilon^2) & 4\sigma^2\epsilon^2 \\ 2\sigma\epsilon(\sigma^2+\epsilon^2) & (\sigma^2+\epsilon^2)^2 & 4\sigma^2\epsilon^2 & 2\sigma\epsilon(\sigma^2+\epsilon^2) \\ 2\sigma\epsilon(\sigma^2+\epsilon^2) & 4\sigma^2\epsilon^2 & (\sigma^2+\epsilon^2)^2 & 2\sigma\epsilon(\sigma^2+\epsilon^2) \\ 4\sigma^2\epsilon^2 & 2\sigma\epsilon(\sigma^2+\epsilon^2) & 2\sigma\epsilon(\sigma^2+\epsilon^2) & (\sigma^2+\epsilon^2)^2 \end{pmatrix}.$$

Corollary 4.3. *The zero solution of the stochastic difference equation generated by the θ-Milstein method applied to the test equation* (28) *is asymptotically mean-square stable if and only if*

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma+|\epsilon|)^2 + \frac{1}{2}h^2(\sigma+|\epsilon|)^4}{(1-\lambda h\theta)^2} < 1\,. \tag{36}$$

Proof. The stability matrix $\mathcal{S}^{\text{Mil}}_{(28)}$ has three distinct eigenvalues, which have been determined using Maple as

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma\pm\epsilon)^2 + \frac{1}{2}h^2(\sigma\pm\epsilon)^4}{(1-\lambda h\theta)^2}$$

and

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2-\epsilon^2) + \frac{1}{2}h^2(\sigma^2-\epsilon^2)^2}{(1-\lambda h\theta)^2}\,,$$

where the maximum of the first two yields (36) based on Lemma 3.4. □

4.5. The stability condition for the θ-Milstein method for the test system with commutative noise

The stability matrix $\mathcal{S}^{\text{Mil}}_{(29)}$ arising from the application of the θ-Milstein method to the test system (29) can be determined via formula (24) and we obtain

$$\begin{aligned}\mathcal{S}^{\text{Mil}}_{(29)} &= (A \otimes A) + (B_1 \otimes B_1) + (B_2 \otimes B_2) + \\ &\quad +2(C_{1,1} \otimes C_{1,1}) + 2(C_{2,2} \otimes C_{2,2}) + ((C_{1,2} + C_{2,1}) \otimes (C_{1,2} + C_{2,1})) \\ &= \begin{pmatrix} a+b_1+c_1+c_2 & 0 & 0 & b_2+c_3 \\ 0 & a+b_1+c_1+c_2 & -b_2-c_3 & 0 \\ 0 & -b_2-c_3 & a+b_1+c_1+c_2 & 0 \\ b_2+c_3 & 0 & 0 & a+b_1+c_1+c_2 \end{pmatrix},\end{aligned}$$

where a, b_1 and b_2 are as in (33) and

$$c_1 = \frac{\frac{1}{2}h^2\sigma^4}{(1-h\theta\lambda)^2}, \quad c_2 = \frac{\frac{1}{2}h^2\epsilon^4}{(1-h\theta\lambda)^2}, \quad \text{and} \quad c_3 = \frac{h^2\sigma^2\epsilon^2}{(1-h\theta\lambda)^2}. \tag{37}$$

Corollary 4.4. *The zero solution of the stochastic difference equation generated by the θ-Milstein method applied to the test equation* (29) *is asymptotically mean-square stable if and only if*

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2+\epsilon^2) + \frac{1}{2}h^2(\sigma^2+\epsilon^2)^2}{(1-\lambda h\theta)^2} < 1. \tag{38}$$

Proof. Using Maple we find that the stability matrix $\mathcal{S}^{\text{Mil}}_{(29)}$ has two double eigenvalues, given by

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2-\epsilon^2) + \frac{1}{2}h^2(\sigma^2-\epsilon^2)^2}{(1-\lambda h\theta)^2}$$

and

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2+\epsilon^2) + \frac{1}{2}h^2(\sigma^2+\epsilon^2)^2}{(1-\lambda h\theta)^2}.$$

By Lemma 3.4, the latter eigenvalue determines the stability condition (38). □

4.6. The stability condition for the θ-Milstein method for the test system with non-commutative noise

Finally, the stability matrix $\mathcal{S}^{\text{Mil}}_{(30)}$, originating from the θ-Milstein method applied to the test system (30), can be determined by formula (27) to be

$$
\begin{aligned}
\mathcal{S}^{\text{Mil}}_{(30)} &= (A\otimes A)+(B_1\otimes B_1)+(B_2\otimes B_2)+2(C_{1,1}\otimes C_{1,1})+2(C_{2,2}\otimes C_{2,2})\\
&+ \big((C_{1,2}+C_{2,1})\otimes(C_{1,2}+C_{2,1})\big)+\Big(\frac{6}{\pi^2}\sum_{k=1}^{p}\frac{1}{k^2}\Big)\big((C_{1,2}-C_{2,1})\otimes(C_{1,2}-C_{2,1})\big)\\
&= \begin{pmatrix} a+b_1+c_1+c_2 & 0 & 0 & b_2+c_3\\ 0 & a-b_1+c_1+c_2 & b_2-c_3 & 0\\ 0 & b_2-c_3 & a-b_1+c_1+c_2 & 0\\ b_2+c_3 & 0 & 0 & a+b_1+c_1+c_2 \end{pmatrix},
\end{aligned}
$$

where a, b_1 and b_2 are as in (33), c_1, c_2 have been defined in (37) and

$$c_3 = \frac{h^2\sigma^2\epsilon^2}{(1-h\theta\lambda)^2}\Big(\frac{6}{\pi^2}\sum_{k=1}^{p}\frac{1}{k^2}\Big).$$

Here, we have used that $\big((C_{1,2}+C_{2,1})\otimes(C_{1,2}+C_{2,1})\big)=0$. When we estimate the finite sum $\sum_{k=1}^{p}1/k^2$ by its limit $\pi^2/6$ for $p\to\infty$, then $c_3 = (h^2\sigma^2\epsilon^2)/(1-h\theta\lambda)^2$. We obtain the following result.

Corollary 4.5. *The zero solution of the stochastic difference equation generated by the θ-Milstein method applied to the test equation* (29) *is asymptotically mean-square stable if and only if*

$$\frac{(1+(1-\theta)h\lambda)^2+h(\sigma^2+\epsilon^2)+\frac{1}{2}h^2(\sigma^2+\epsilon^2)^2+(K(p)-1)h^2\sigma^2\epsilon^2}{(1-\lambda h\theta)^2}<1\,. \tag{39}$$

where $K(p):=\frac{6}{\pi^2}\sum_{k=1}^{p}\frac{1}{k^2}$ *with* $K(p)\to 1$ *for* $p\to\infty$.

Remark 4.6. *Taking the limit of $K(p)$ for $p\to\infty$ essentially corresponds to using the exact representation of the Lévy stochastic area integrals* (25)*, then condition* (39) *is identical to condition* (38)*. This is in accordance with the observation made in [6] that when considering mean-square stability for linear systems and their numerical approximations, the influence of the diffusion terms on the stability properties is due to the noise intensity rather than the structure of the diffusion matrices.*

Proof. : The stability matrix $\mathcal{S}^{\mathrm{Mil}}_{(30)}$ has four distinct eigenvalues, derived with MAPLE as

$$\frac{(1+(1-\theta)h\lambda)^2 + h(\sigma^2 \pm \epsilon^2) + \frac{1}{2}h^2(\sigma^2+\epsilon^2)^2 + (\pm K(p)-1)h^2\sigma^2\epsilon^2}{(1-\lambda h\theta)^2}$$

and

$$\frac{(1+(1-\theta)h\lambda)^2 + h(-\sigma^2 \pm \epsilon^2) + \frac{1}{2}h^2(\sigma^2+\epsilon^2)^2 + (\mp K(p)-1)h^2\sigma^2\epsilon^2}{(1-\lambda h\theta)^2}\,.$$

By Lemma 3.4 the stability condition (39) is determined by the maximum of the first two eigenvalues. □

4.7. Comparison of the stability conditions

A standard approach to visually compare the stability properties of numerical methods consists of plotting stability regions described by the boundaries of the sets of parameters of the test equations as well as the numerical methods, such that for all parameters within these boundaries the corresponding stability conditions are satisfied. An advantage of choosing test systems of SDEs containing only a few parameters is that it is possible to plot and compare the mean-square stability regions in three-dimensional plots, as in the case of (28) to (30) there are only three parameters involved. In order to have the stability conditions (31) and (32) for the test SDE systems and the conditions for the numerical methods in a comparable format, we rearrange the latter into the following equivalent versions.

Corollary 4.7. *The zero-solution of the stochastic difference equation generated by the θ-Maruyama method applied to the test-equations is asymptotically mean-square stable if and only if*

$$\text{for (28) (cf. (34)): } \lambda + \frac{1}{2}(\sigma^2+\epsilon^2+2|\sigma\epsilon|) + \frac{1}{2}h(1-2\theta)\lambda^2 < 0\,, \tag{40}$$

$$\text{for (29) and (30) (cf. (35)): } \lambda + \frac{1}{2}(\sigma^2+\epsilon^2) + \frac{1}{2}h(1-2\theta)\lambda^2 < 0\,. \tag{41}$$

The zero solution of the stochastic difference equation generated by the θ-Milstein method applied to the test equations is asymptotically mean-square *stable if and only if*

$$\text{for (28) (cf. (36)):} \quad \lambda + \frac{1}{2}(\sigma + |\epsilon|)^2 + \frac{1}{2}h(1-2\theta)\lambda^2 + \frac{1}{4}h(\sigma + |\epsilon|)^4 < 0\,, \tag{42}$$

$$\text{for (29) (cf. (38)):} \quad \lambda + \frac{1}{2}(\sigma^2 + \epsilon^2) + \frac{1}{2}h(1-2\theta)\lambda^2 + \frac{1}{4}h(\sigma^2 + \epsilon^2)^2 < 0\,, \tag{43}$$

and for (30) (cf. (39)):

$$\lambda + \frac{1}{2}(\sigma^2 + \epsilon^2) + \frac{1}{2}h(1-2\theta)\lambda^2 + \frac{1}{4}h(\sigma^2 + \epsilon^2)^2 + (K(p)-1)h\sigma^2\epsilon^2 < 0\,. \tag{44}$$

Following [18] regarding the scaling of the parameters in the plots we now set

$$x := h\lambda, \quad y := h\sigma^2 \quad \text{and} \quad z := h\epsilon^2\,.$$

The stability conditions (31), (32) for the test SDEs and (40) to (44) for the numerical methods in terms of x, y and z with $x \in \mathbb{R}$ and y, $z \in \mathbb{R}^+$ are then given by the following inequalities.

(31) reads:	$x + \frac{1}{2}y + \frac{1}{2}z + \sqrt{y\,z}$	$<$	$0\,;$
(32) reads:	$x + \frac{1}{2}y + \frac{1}{2}z$	$<$	$0\,;$
(40) reads:	$x + \frac{1}{2}(\sqrt{y} + \sqrt{z})^2 + \frac{1}{2}(1-2\theta)x^2$	$<$	$0\,;$
(41) reads:	$x + \frac{1}{2}(y+z) + \frac{1}{2}(1-2\theta)x^2$	$<$	$0\,;$
(42) reads:	$x + \frac{1}{2}(\sqrt{y} + \sqrt{z})^2 + \frac{1}{2}(1-2\theta)x^2 + \frac{1}{4}(\sqrt{y} + \sqrt{z})^4$	$<$	$0\,;$
(43) reads:	$x + \frac{1}{2}(y+z) + \frac{1}{2}(1-2\theta)x^2 + \frac{1}{4}(y^2 + 2yz + z^2)$	$<$	$0\,;$
(44) reads:	$x + \frac{1}{2}(y+z) + \frac{1}{2}(1-2\theta)x^2 + \frac{1}{4}(y^2 + 2yz + z^2) + (K(p)-1)yz$	$<$	$0\,.$

Figures 1 and 2 illustrate the different stability properties of the two methods applied to the different test systems. Figure 2 includes the representation of the stability regions for the θ-Milstein method applied to the test system (30) in the limit to infinity of the parameter p in the approximation of the stochastic Lévy area. In this case the conditions (43) and (44) coincide. We have performed further experiments and plotted analogous pictures for the θ-Milstein method applied to the test system (30) for other values of p with no visible difference to the picture in Figure 2. We thus observe

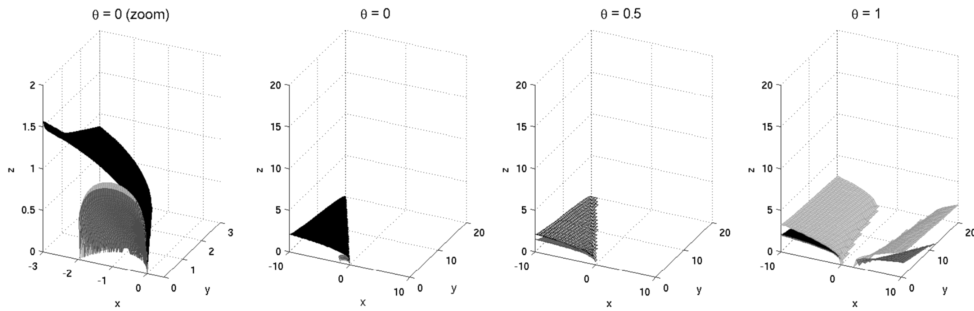

Figure 1: Boundaries of the mean-square stability regions for the test system (28) (black area) and the θ-Maruyama (light grey area) and the θ-Milstein method (dark grey area) for $\theta = 0, 0.5, 1$. The first plot provides a zoom into the second plot, where $\theta = 0$. For $\theta = 0.5$ the stability regions of (28) and the θ-Maruyama method coincide.

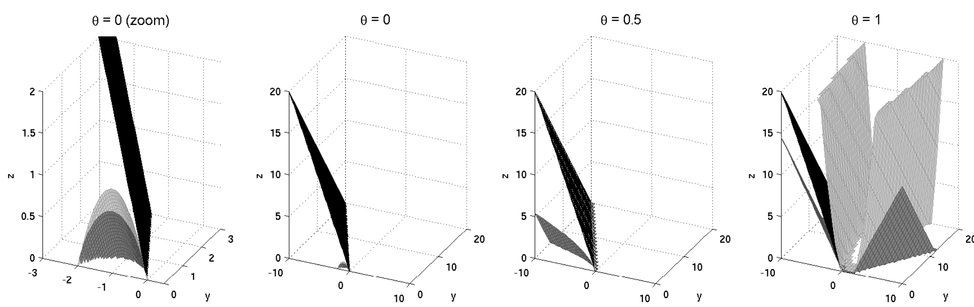

Figure 2: Boundaries of the mean-square stability regions for the test systems (29) and (30) (black area) and the θ-Maruyama (light grey area) and the θ-Milstein method (dark grey area) for $\theta = 0, 0.5, 1$. The first plot provides a zoom into the second plot, where $\theta = 0$. For $\theta = 0.5$ the stability regions for (29) and (30) and the θ-Maruyama method coincide.

that the mean-square stability regions for the θ-Milstein method are not significantly affected by the additional approximation of the stochastic Lévy area. However they are always smaller than the stability region of the θ-Maruyama method. As reported in [8, 18] in the case of scalar SDEs with one or several driving Wiener processes, the mean-square stability regions of the θ-Maruyama method enclose that of the SDE if $\theta \geq 1/2$.

As a further aspect of providing comparisons of the stability properties of the methods we now consider a fixed set of values of the parameters in Equations (28) to (30) and, using MATLAB, numerically evaluate the spectral abscissa of the stability matrix S and the spectral radius of the stability matrices $\mathcal{S}$ for different step-sizes h and method parameter θ. Tables 1 to 3 provide the corresponding values and illustrate clearly the different stability properties of the two methods considered. Although we here continue to use the test systems (28) to (30), we emphasise that these investigations can be made with other test systems having more parameters and other Maruyama- and Milstein-type methods, such as in Example 3.12, and MATLAB files to perform these calculations can be downloaded via `http://www.jku.at/stochastik/content/e140916/` in the folder *Software* of the first author.

We complement these numerical results with simulation studies. Figures 3 to 5 show the estimated mean-square norm of the first component X_ν of the numerical solution X, which is estimated pointwise by

$$\mathsf{E}((X_\nu(t_i))^2)^{1/2} \approx \left(\frac{1}{M} \sum_{j=1}^{M} (X_{\nu,i}(\omega_j))^2 \right)^{1/2}, \ \nu = 1, 2 \, .$$

For the evaluation we used $M = 1000000$ sample paths. Although the number of paths is quite high, the fact that the variance of the estimator above grows with time impinges upon the smoothness of the calculated mean-square norms.

5. Conclusion

We have developed a structural approach for a linear mean-square stability analysis of systems of stochastic differential equations with several multiplicative noise terms as well as of numerical methods for their approximation. In particular, we propose the use of the vectorisation of matrices and the

step-size	θ-Maruyama approximation			θ-Milstein approximation		
	$\theta = 0$	$\theta = 0.5$	$\theta = 1$	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
h=1.00	8.982 unst.	0.837 stable	0.374 stable	21.39 unst.	2.823 unst.	1.150 unst.
h=0.50	2.741 unst.	0.834 stable	0.559 stable	5.844 unst.	1.850 unst.	1.060 unst.
h=0.20	1.156 unst.	0.880 stable	0.780 stable	1.653 unst.	1.173 unst.	0.974 stable
h=0.10	0.990 stable	0.923 stable	0.887 stable	1.112 unst.	1.017 unst.	0.960 stable
h=0.05	0.972 stable	0.956 stable	0.945 stable	1.003 unst.	0.983 stable	0.968 stable
h=0.02	0.983 stable	0.981 stable	0.979 stable	0.988 stable	0.986 stable	0.983 stable
h=0.01	0.991 stable	0.990 stable	0.990 stable	0.992 stable	0.991 stable	0.991 stable

Table 1: Values of the spectral radius of the stability matrices $\mathcal{S}^{\text{Mar}}_{(28)}$ and $\mathcal{S}^{\text{Mil}}_{(28)}$, corresponding to the stability inequalities (34) and (36), respectively, for the test-equation (28) with $\lambda = -3$, $\sigma = 0.5$, $\epsilon = \sqrt{3}$. The spectral abscissa of the stability matrix $S_{(28)}$ is -1.018.

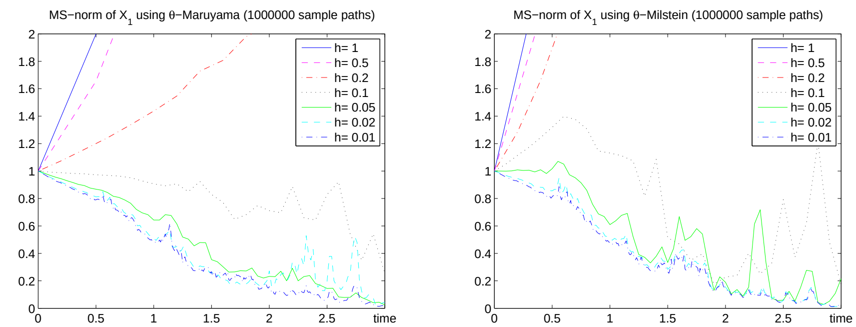

Figure 3: Estimated mean-square norm of the first component of the iterates of the θ-Maruyama and θ-Milstein method applied to (28), for 1000000 sample paths, $\theta = 0$ and different values of the step-sizes.

step-size	θ-Maruyama approximation			θ-Milstein approximation		
	$\theta = 0$	$\theta = 0.5$	$\theta = 1$	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
h=1.00	4.250 unst.	0.813 stable	0.472 stable	9.531 unst.	2.133 unst.	1.059 unst.
h=0.50	1.625 unst.	0.833 stable	0.656 stable	2.945 unst.	1.420 unst.	0.986 stable
h=0.20	1.010 unst.	0.896 stable	0.842 stable	1.221 unst.	1.043 unst.	0.950 stable
h=0.10	0.965 stable	0.938 stable	0.920 stable	1.018 unst.	0.982 stable	0.957 stable
h=0.05	0.973 stable	0.966 stable	0.961 stable	0.986 stable	0.978 stable	0.972 stable
h=0.02	0.987 stable	0.986 stable	0.985 stable	0.989 stable	0.988 stable	0.987 stable
h=0.01	0.993 stable	0.993 stable	0.992 stable	0.993 stable	0.993 stable	0.993 stable

Table 2: Values of the spectral radius of the stability matrices $\mathcal{S}^{\text{Mar}}_{(29)}$ and $\mathcal{S}^{\text{Mil}}_{(29)}$, corresponding to the stability inequalities (35) and (38), respectively, for the test-equation (29) with $\lambda = -2$, $\sigma = 0.5$, $\epsilon = \sqrt{3}$. The spectral abscissa of the stability matrix $S_{(29)}$ is -0.750.

step-size	θ-Maruyama approximation			θ-Milstein approximation			
	$\theta = 0$	$\theta = 0.5$	$\theta = 1$	$p = 1/h$	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
h=1	4.250 unst.	0.813 stable	0.472 stable	p= 1	9.237 unst.	2.059 unst.	1.026 unst.
h=0.5	1.625 unst.	0.833 stable	0.656 stable	p= 2	2.900 unst.	1.400 unst.	0.975 stable
h=0.2	1.010 unst.	0.896 stable	0.842 stable	p= 5	1.218 unst.	1.040 unst.	0.948 stable
h=0.10	0.965 stable	0.938 stable	0.920 stable	p= 10	1.017 unst.	0.981 stable	0.957 stable
h=0.05	0.973 stable	0.966 stable	0.961 stable	p= 20	0.986 stable	0.978 stable	0.972 stable
h=0.02	0.987 stable	0.986 stable	0.985 stable	p= 50	0.989 stable	0.988 stable	0.987 stable
h=0.01	0.993 stable	0.993 stable	0.992 stable	p= 100	0.993 stable	0.993 stable	0.993 stable

Table 3: Values of the spectral radius of the stability matrices $\mathcal{S}^{\text{Mar}}_{(30)}$ and $\mathcal{S}^{\text{Mil}}_{(30)}$ the stability inequalities (35) and (39), respectively, for the test-equation (30) with $\lambda = -2$, $\sigma = 0.5$, $\epsilon = \sqrt{3}$. The spectral abscissa of the stability matrix $S_{(30)}$ is -0.750.

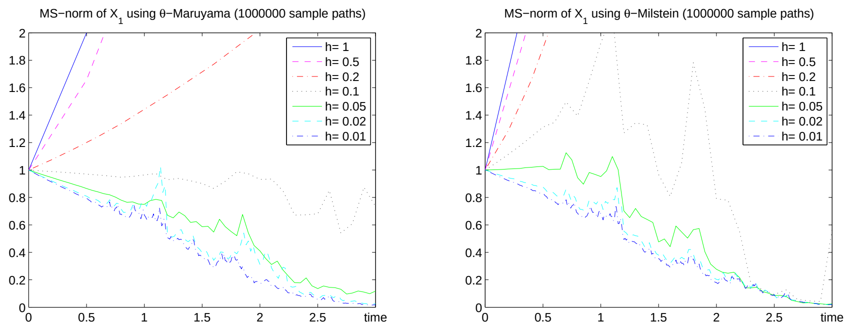

Figure 4: Estimated mean-square norm of the first component of the iterates of the θ-Maruyama and θ-Milstein method applied to (29), for 1000000 sample paths, $\theta = 0$ and different values of the step-sizes.

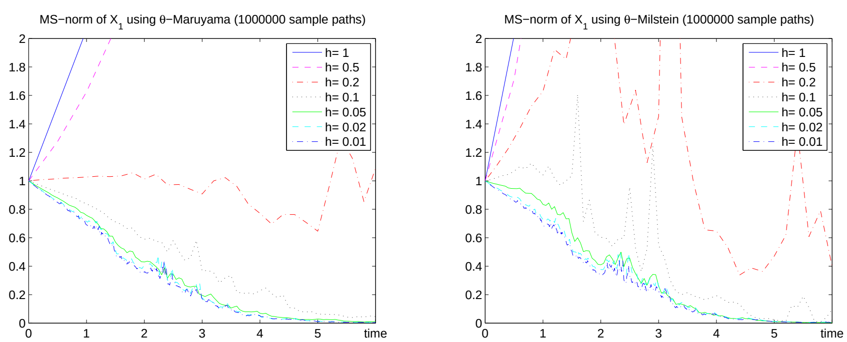

Figure 5: Estimated mean-square norm of the first component of the iterates of the θ-Maruyama and θ-Milstein method applied to (30), for 1000000 sample paths, $\theta = 0$ and different values of the step-sizes.

Kronecker product to treat the multi-dimensional systems involved in this analysis. The methods we considered are the θ-Maruyama and the θ-Milstein method, and we have first derived the structure of the stability matrices arising for their application to systems of SDEs with different noise structures. To the best of our knowledge this article contains the first stability analysis of the θ-Milstein method applied to systems of SDEs with non-commutative noise terms. In the case of a general SDE (2) the number of parameters is too large to derive explicitly stability conditions that provide any meaningful insight, however, given numerical values for the parameters, it is easy to check via MATLAB if the stability conditions are satisfied for any of the methods and a chosen step-size. MATLAB files to perform these calculations can be downloaded via `http://www.jku.at/stochastik/content/e140916/` in the folder *Software* of the first author. We then have chosen appropriate test SDE systems involving only a small number of parameters but with three different noise structures representing three cases were the θ-Milstein method involves different implementations of the double Wiener integral. These were used to compare the stability properties of the θ-Maruyama and θ-Milstein method. We have found that, on the whole, the θ-Maruyama method is quite robust in terms of mean-square stability with respect to changes in the noise structure. This is consistent with the results in [8], but differs from the case of almost sure stability, see [6]. The mean-square stability analysis for the θ-Milstein method yields much more restrictive stability conditions than those for the θ-Maruyama method (cf. again [8]), and in addition these depend on the noise structure and the implementations of the double Wiener integral as well. As a positive result, we report that in the case of the test SDE system (30) with non-commutative noise, although the stability condition for the θ-Milstein method contains another positive term due to the approximation of the Lévy areas, its effect appears to be negligible.
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