Efficient transient noise analysis in circuit simulation

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Stochastic differential algebraic equations (SDAEs) arise as a mathematical model for electrical network equations that are influenced by additional sources of Gaussian white noise.
We discuss adaptive linear multi-step methods for their numerical integration, in particular stochastic analogues of the trapezoidal rule and the two-step backward differentiation formula, and we obtain conditions that ensure mean-square convergence of this methods. For the case of small noise we present a strategy for controlling the step-size in the numerical integration. It is based on estimating the mean-square local errors and leads to step-size sequences that are identical for all computed paths. Test results illustrate the performance of the presented methods.

1 Transient noise analysis in circuit simulation

The increasing scale of integration, high tact frequencies and low supply voltages cause smaller signal-to-noise ratios. In several applications the noise influences the system behavior in an essentially nonlinear way such that linear noise analysis is no longer satisfactory and transient noise analysis, i.e. the simulation of noisy systems in the time domain, becomes necessary. We deal with the thermal noise of resistors as well as the shot noise of semiconductors that are modelled by additional sources of additive or multiplicative Gaussian white noise currents.

Thermal noise of resistors having a resistance R, is caused by the thermal motion of electrons and is modelled by additive noise,

$$i_{th} = \sqrt{\frac{2kT}{R}}\xi(t), \qquad k = 1.3806 \times 10^{-23},$$

where T is the temperature, k is Boltzmann's constant and $\xi(t)$ is a standard Gaussian white noise process. Shot noise of pn-junctions, caused by the discrete nature of currents due to the elementary charge, is modelled by multiplicative noise,

$$i_{shot} = \sqrt{q_e|g(u)|}\xi(t), \qquad q_e = 1.602 \times 10^{-19},$$

where $g(u)$ is the noisefree current for the voltage u and q_e is the elementary charge. Combining Kirchhoff's Current law with the element characteristics and using the charge-oriented formulation yields a stochastic differential algebraic equation (SDAE) of the form

$$A\frac{d}{dt}q(X(t)) + f(X(t),t) + \sum_{r=1}^{m} g_r(X(t),t)\xi_r(t) = 0\,, \qquad t \in [t_0, T]\,, \tag{1}$$

where A is a constant singular matrix determined by the topology of the electrical network, the vector $q(x)$ consists of the charges and the fluxes, and x is the vector of unknowns consisting of the nodal potentials and the branch currents. The term $f(x,t)$ describes the impact of the static elements, $g_r(x,t)$ denotes the vector of noise intensities for the r-th noise source, and ξ is an m-dimensional vector of independent Gaussian white noise sources (see e.g. [2, 6, 7]). One has to deal with a large number of equations as well as of noise sources. Compared to the other quantities the noise intensities $g_r(x,t)$ are small.

We understand (1) as a stochastic integral equation

$$Aq(X(s))|_{t_0}^{t} + \int_{t_0}^{t} f(X(s),s)ds + \sum_{r=1}^{m}\int_{t_0}^{t} g_r(X(s),s)dW_r(s) = 0\,, \qquad t \in [t_0, T]\,, \tag{2}$$

where the second integral is an Itô-integral, and W denotes an m-dimensional Wiener process (or Brownian motion) given on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t \geq t_0}$. The solution is a stochastic process depending on the time t and on the random sample ω. Typical paths are nowhere differentiable. Using techniques from the theory of DAEs as well as of the theory of stochastic differential equations (SDEs) one derives existence and uniqueness for the solutions as well as convergence results for certain drift-implicit methods for systems with DAE-index 1 [7].

Especially for oscillating solutions in circuit simulation one is interested in the phase noise. We therefore aim at the simulation of solution paths that reveal the phase noise and apply the concept of strong solutions and strong (mean-square) convergence of approximations.

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2 Numerical Methods

We present adaptations of known schemes for SDEs that are implicit in the deterministic part (drift) and explicit in the stochastic part (diffusion) of the SDAE. Designing the methods such that the iterates have to fulfill the constraints of the SDAE at the current time-point is the key idea to adapt known methods for the SDEs to (2). This is realized by suitable implicit discretizations of the deterministic part (drift). We consider stochastic analogues of the two-step backward differentiation formula (BDF_2) and the trapezoidal rule, where only the increments of the driving Wiener process are used to discretize the diffusion part. Analogously to the Euler-Maruyama scheme we call such methods multi-step Maruyama methods. The BDF_2 Maruyama method for the SDAE (2) has the form [1, 2, 4, 6]

$$A\frac{q(X_\ell)-\frac{(\kappa_\ell+1)^2}{2\kappa_\ell+1}q(X_{\ell-1})+\frac{\kappa_\ell^2}{2\kappa_\ell+1}q(X_{\ell-2})}{h_\ell}+\frac{\kappa_\ell+1}{2\kappa_\ell+1}f(X_\ell,t_\ell)$$
$$+\sum_{r=1}^{m}g_r(X_{\ell-1},t_{\ell-1})\,\frac{\Delta W_r^\ell}{h_\ell}-\frac{\kappa_\ell^2}{2\kappa_\ell+1}\sum_{r=1}^{m}g_r(X_{\ell-2},t_{\ell-2})\,\frac{\Delta W_r^{\ell-1}}{h_\ell}=0\,,\quad \ell=2,\dots,N. \tag{3}$$

Here, X_ℓ denotes the approximation to $X(t_\ell)$, $h_\ell=t_\ell-t_{\ell-1}$, and $\Delta W_r^\ell=W_r(t_\ell)-W_r(t_{\ell-1})\sim N(0,h_\ell)$ on the grid $0=t_0<t_1<\cdots<t_N=T$. The coefficients of the two-step scheme (3) depend on the step-size ratio $\kappa_\ell=h_\ell/h_{\ell-1}$ and have to satisfy the conditions for consistency of order one and two in the deterministic case and of order $1/2$ in the stochastic case.

A correct formulation of the stochastic trapezoidal rule for SDAEs requires more structural information. It should implicitly realize the stochastic trapezoidal rule for the so called inherent regular SDE of (2) that governs the dynamical components (cf. [7]). However, neither the straightforward approach

$$A\frac{q(X_\ell)-q(X_{\ell-1})}{h_\ell}+\frac{1}{2}\big(f(X_\ell,t_\ell)+f(X_{\ell-1},t_{\ell-1})\big)+\sum_{r=1}^{m}g_r(X_{\ell-1},t_{\ell-1})\frac{\Delta W_r^\ell}{h_\ell}=0,\quad \ell=1,\dots,N, \tag{4}$$

nor the naive extension of the usual formulation for DAEs to SDAEs (cf. [7])

$$A\Big(-Y_{\ell-1}+2\frac{q(X_\ell)-q(X_{\ell-1})}{h_\ell}\Big)+f(X_\ell,t_\ell)+\sum_{r=1}^{m}g_r(X_{\ell-1},t_{\ell-1})\frac{\Delta W_r^\ell}{h_\ell}=0,\qquad Y_\ell:=-Y_{\ell-1}+2\frac{q(X_\ell)-q(X_{\ell-1})}{h_\ell} \tag{5}$$

work correctly. To explain this we look at the constraints of the SDAE (2) and the numerical schemes (4),(5). Let R denote a projector along $\mathrm{im}A$. Then one has $RA=0$ and the constraints of (2) are given by $Rf(X(t),t)+\sum_{r=1}^m Rg_r(X(t),t)\xi_r(t)=0$. The SDAE is called *without direct noise* if the constraints are noise-free, i.e., $Rg_r=0,\ r=1,\dots,m$, otherwise the SDAE is called *with direct noise*. In the first scheme (4) the constraints are not correctly discretized. For noise-free constraints one has $Rf(X_\ell,t_\ell)=-Rf(X_{\ell-1},t_{\ell-1})$ instead of $Rf(X_\ell,t_\ell)=0$ such that inconsistencies in the constraints are never corrected. For SDAEs with direct noise the situation is worse, since the noise in the constraints is accumulated to

$$Rf(X_\ell,t_\ell)=(-1)^\ell Rf(X_0,t_0)+2\sum_{j=1}^{\ell}(-1)^j\sum_{r=1}^{m}Rg_r(X_{\ell-j},t_{\ell-j})\frac{\Delta W_r^{\ell-j+1}}{h_{\ell-j+1}},\quad \ell=1,\dots,N. \tag{6}$$

Using the scheme (5), the constraints are correctly discretized, however the discretization of the inherent SDE is incorrect in the diffusion part. Since one has $A(Y_{\ell-1})=-f(X_{\ell-1},t_{\ell-1})-\sum_{r=1}^m g_r(X_{\ell-2},t_{\ell-2})\frac{\Delta W_r^{\ell-1}}{h_{\ell-1}}$, the scheme (5) leads to the inconsistent discretization

$$A\frac{q(X_\ell)-q(X_{\ell-1})}{h_\ell}+\frac{1}{2}\big(f(X_\ell,t_\ell)+f(X_{\ell-1},t_{\ell-1})\big)+\sum_{r=1}^{m}\frac{1}{2}\Big(g_r(X_{\ell-1},t_{\ell-1})\frac{\Delta W_r^\ell}{h_\ell}+g_r(X_{\ell-2},t_{\ell-2})\frac{\Delta W_r^{\ell-1}}{h_{\ell-1}}\Big)=0.$$

A possible way out is to discretize the constraints differently, which requires the explicit knowledge of the constraints or, equivalently, a projector R along $\mathrm{im}A$. The discrete equations

$$A\frac{q(X_\ell)-q(X_{\ell-1})}{h_\ell}+\frac{1}{2}(I-R)\big(f(X_\ell,t_\ell)+f(X_{\ell-1},t_{\ell-1})\big)+Rf(X_\ell,t_\ell)+\sum_{r=1}^{m}g_r(X_{\ell-1},t_{\ell-1})\frac{\Delta W_r^\ell}{h_\ell}=0, \tag{7}$$

$\ell=1,\dots,N$, imply the correct constraints and realize the trapezoidal rule for the inherent regular SDE (cf. [7]).

Both the BDF_2 (3) and the trapezoidal rule (7) have only an asymptotic order of strong convergence of $1/2$, i.e. ,

$$\|X(t_\ell)-X_\ell\|_{L_2(\Omega)}:=\max_{\ell=1,\dots,N}(E|X(t_\ell)-X_\ell|^2)^{1/2}\le c\cdot h^{1/2}, \tag{8}$$

where $h := \max_{\ell=1,\ldots,N} h_\ell$ is the maximal step-size of the grid. (For additive noise the order may be 1.) This holds true, in general, for all numerical schemes that include only information on the increments of the Wiener process. However, when the noise is small, the error behavior is much better. In fact, the errors are dominated by the deterministic terms as long as the step-size is large enough [1]. In more detail, the error of both methods is bounded by $O(h^2 + \varepsilon h + \varepsilon^2 h^{1/2})$, when ε is used to measure the smallness of the noise $(g_r(x,t) = \varepsilon \hat{g}_r(x,t),\ r = 1,\ldots,m,\ \varepsilon \ll 1)$. Thus, we can expect order 2 behavior if $h \gg \varepsilon$.

The smallness of the noise also allows special estimates of the local error terms, which can be used to control the step-size. In [3] the authors presented a step-size control for the drift-implicit Euler scheme in the case of small noise that leads to adaptive step-size sequences that are uniform for all paths, see also [2, 6]. The estimates of the dominating local error term are based on values of the drift term and do not cost additional evaluations of the coefficients of the SDE or their derivatives. In [5] we present an error estimate and, based on this, a step-size control for stochastic linear multi-step methods with deterministic order 2.

3 Numerical results

Here, we illustrate the potential of the step-size control strategy by simulation results for the stochastic BDF_2 applied to two test problems. First, we consider a nonlinear scalar SDE with known explicit solution. The drift and diffusion coefficients are tunable by real parameters α and β, which we have chosen as $\alpha = -10$ and $\beta = 0.01$:

$$X(t) = \int_0^t -(\alpha + \beta^2 X(s))(1 - X(s)^2)ds + \int_0^t \beta(1 - X(s)^2)dW(s), \quad t \in [0,T]\,. \tag{9}$$

The solution is given by

$$X(t) = \frac{\exp(-2\alpha t + 2\beta W(t)) - 1}{\exp(-2\alpha t + 2\beta W(t)) + 1}\,. \tag{10}$$

In Figure 1 we present a work-precision diagram. We plotted the tolerance ($\triangle$) and the mean-square norm of the errors for adaptively chosen (+) and constant ($\times$) step-sizes for 100 computed paths vs. the number of steps in logarithmic scale. Lines with slopes -2 and -0.5 are provided to enable comparisons with convergence of order 2 or $1/2$. We observe order 2 behavior up to accuracies of 10^{-4}. The results show that the proposed step-size control works very well for step-sizes above this threshold.

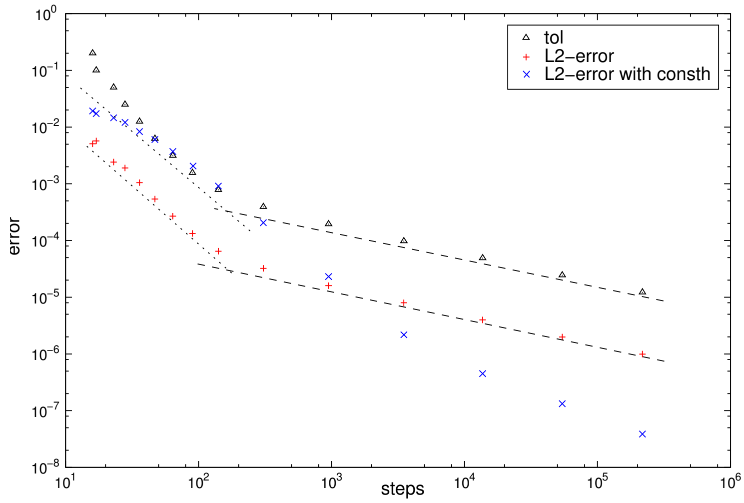

Fig. 1 Tolerance and accuracy versus steps for a test-SDE.

Secondly, we consider a model of an inverter circuit with a MOSFET-transistor under the influence of thermal noise. The equivalent circuit diagram is given in the left picture of Figure 2. The MOSFET is modelled as a current source from source to drain that is controlled by the nodal potentials at gate, source and drain. The thermal noise of the resistor and of the MOSFET is modelled by additional white noise current sources that are shunt in parallel to the original, noise-free elements. To make the effect of the noise more visible we scaled the noise intensities by a factor of 1000. For the simulation we used the BDF_2 with adaptively chosen stepsizes using the information of 100 simultaneously computed paths.

In the right picture of Figure 2 we plotted the input voltage U_{in} and values of the output voltage e_1 versus time. The dark lines show the values of two different solution paths, the dotted line gives the mean of 100 paths and the gray lines the

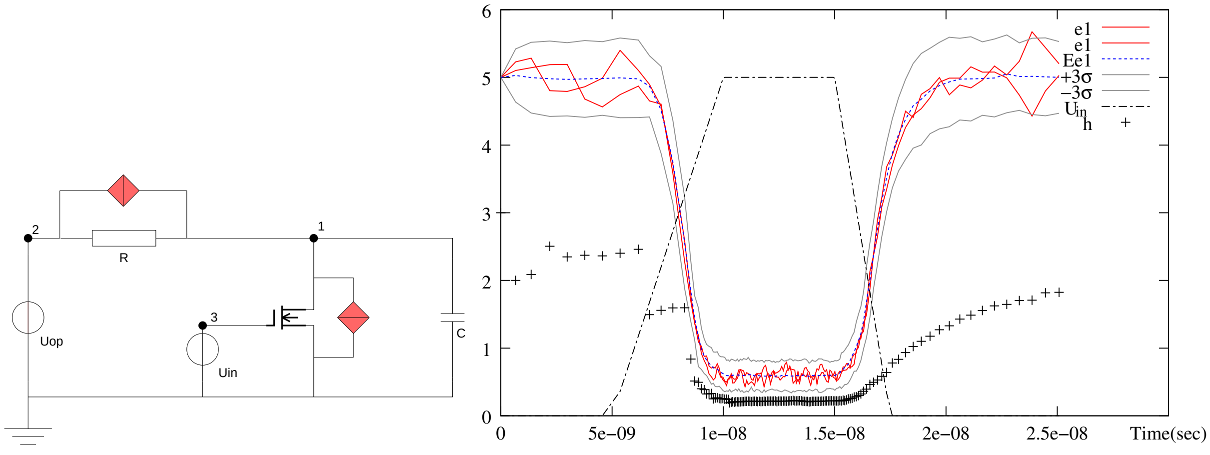

Fig. 2 Thermal noise sources in a MOSFET inverter circuit: Circuit diagram (left) and simulation results (right).

3σ-confidence interval for the output voltage e_1. Moreover, the applied step-sizes, suitably scaled, are shown by means of single crosses.

Using the information of an ensemble of simultaneously computed solution paths smoothes the step-size sequence and reduces the number of rejected steps considerably, compared to the simulation of a single path.

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