Time-continuous Production Networks with Random Breakdowns

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joint work with
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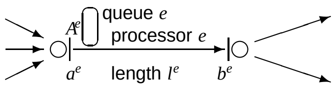
Time-continuous modelling

Directed graph used as network model:

The network is modelled as a directed graph where each arc represents a processor and each vertex is a distribution point. Queues in front of processors acting as buffing zones. The density of goods in the processors and the queue load are the quantities of interest.



Densities of the processors and loads of queues:

Density $\rho^e(x,t)$ of processor e transported with velocity $f^e(\rho^e)$ modelled by

$$\text{(PDE)} \qquad \partial_t \rho^e(x,t) + \partial_x f^e(\rho^e(x,t)) = 0\,, \quad x \in [a^e, b^e],\ t \in [t_0, T]\,.$$

Relation between flux and density given by flux function
(v^e is constant transport velocity, μ^e maximal processing rate)

$$f^e(\rho^e) = \min(v^e \cdot \rho^e, \mu^e)\,.$$

Load $q^e(t)$ of the buffing zone (queue) for processor e described by

$$\text{(ODE)} \qquad \partial_t q^e(t) = g^e_{\text{in}}(t) - g^e_{\text{out}}(t)\,, \quad t \in [t_0, T].$$

PDE-ODE coupling:

▷ Ingoing flux function ($A^{s(e),e}$ distribution rate, $\delta^-_{s(e)}$ set of ingoing arcs, $G^{s(e)}_{\text{in}}$ external inflow function)

$$g^e_{\text{in}}(t) = \begin{cases} A^{s(e),e}(t) \sum_{\bar{e} \in \delta^-_{s(e)}} f^{\bar{e}}(\rho^{\bar{e}}(b^{\bar{e}}, t))\,, & \text{if } s(e) \notin \mathcal{V}_{\text{in}}, \\ G^{s(e)}_{\text{in}}(t)\,, & \text{if } s(e) \in \mathcal{V}_{\text{in}}. \end{cases}$$

▷ Outgoing flux function (μ^e maximal processing rate, q^e queue load

$$g^e_{\text{out}}(t) = \begin{cases} \min\{g^e_{\text{in}}(t), \mu^e\}\,, & \text{if } q^e(t) = 0, \\ \mu^e\,, & \text{if } q^e(t) > 0. \end{cases}$$

▷ PDE boundary condition

$$\text{(PDE b.c.)} \qquad \rho^e(a^e, t) = g^e_{\text{out}}(q^e(t))/v^e\,.$$

Breakdown of processors

Single processor:

Modelled by a stochastic two-state process

$$r^e : \mathbb{R}^+_0 \times \Omega \to \{0,1\} : t \times \omega \mapsto r^e(t,\omega)\,,$$

where the time length $\Delta\tau^e$ up to the next switching at $t^* + \Delta\tau^e$ is exponential distributed with rate parameter (Here, τ^e_{on} mean time between failures, and τ^e_{off}: mean repair time.)

$$\lambda(r^e(t^*)) = \begin{cases} 1/\tau^e_{\text{on}}\,, & \text{if } r^e(t^*) = 1, \\ 1/\tau^e_{\text{off}}\,, & \text{if } r^e(t^*) = 0. \end{cases}$$

Including the random breakdowns in the network model:

Replacing the maximal processoing rate μ^e by the time-dependent function

$$\mu^e(t) := \mu^e \cdot r^e(t)$$

▷ $r^e(t) = 0$ (in-operating) $\Rightarrow \mu^e(t) = 0$ and $f^e(\rho^e) = 0$ (no transport)
▷ $r^e(t) = 1$ (operating) $\Rightarrow \mu^e(t) = \mu^e$ and $f^e(\rho^e) = \min(v^e \cdot \rho^e, \mu^e)$ (det. transport)

r^e, ρ^e, and q^e are stochastic processes, but r^e is a random constant between switching times.

▷ The solution $(\rho^1, \ldots, \rho^M, q^1, \ldots, q^M)$ of the network is a piecewise deterministic stochastic process.

Numerics for the coupled system

Sample switching times:

Consider the minimum of all switching times $\Delta\tau = \min\{\Delta\tau^1, \ldots, \Delta\tau^M\}$, that is the length of the time interval $[t^*_i, t^*_{i+1}]$, where all processes r^e are random constants and the network behaves deterministically.

▷ The length of the time interval $\Delta\tau$ until the next switching occurs is exponentially distributed with rate parameter $\lambda^{sum}(t^*_i) = \sum_e \lambda^e(r^e(t^*_i))$.

▷ The index $\bar{e}$ of the next switching processor is proportionally distributed to $\lambda^e(r^e(t^*_i))/\lambda^{sum}(t^*_i)$.

Use deterministic numerics between two switching times t^*_i and t^*_{i+1}:

(Here, $t^*_i = t_C < \cdots < t_j < t_{j+1} < \cdots < t_C = t^*_{i+1}$, and all t_j belong to a global equidistant grid.)

▷ Upwind scheme for the PDE ($\Delta^e x$: constant step-size in space)

$$\bar{\rho}^e(x_{k+1}, t_{j+1}) = \bar{\rho}^e(x_{k+1}, t_j) - \frac{(v^e \cdot \bar{r}^e(t_j))\Delta t_j}{\Delta^e x}\left(\bar{\rho}^e(x_{k+1}, t_j) - \bar{\rho}^e(x_k, t_j)\right)$$

▷ Euler step for the ODE (Δt_j: variable step-size in time)

$$\bar{q}^e(t_{j+1}) = \bar{q}^e(t_j) + \Delta t_j \cdot (g^e_{\text{in}} - g^e_{\text{out}})$$

Repeat this procedure iteratively until the time horizon T is reached.

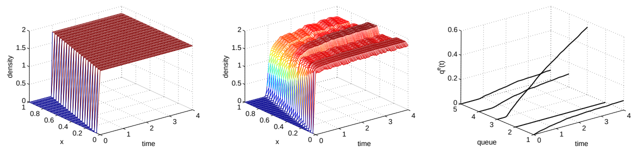
Simulation experiments

A chain of 5 processors

Parameters: 5 processors ($e = 1, \ldots, 5$), processing rates $\mu^e = 2$, velocities $v^e = 1$, lengths $l^e = 0.2$, constant network inflow $G^{s(1)}_{\text{in}}(t) = 1.8$, time horizon $T = 4$, and different mean time between failures (MTBF) and mean repair times (MRT).

processor e	1	2	3	4	5
MTBF τ^e_{on}	0.95	–	0.85	1.90	0.95
MRT τ^e_{off}	0.05	–	0.15	0.10	0.05

Simulation results:



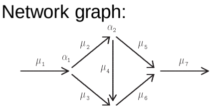
Deterministic density, stochastic density and queue-loads using 1000 scenarios.

A diamond network

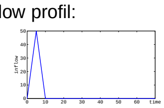
Parameters: 7 processors and 2 controls (α_1, α_2) velocities $v^e = 1$, lengths $l^e = 1$, time horizon $T = 70$.

processor e	μ^e	τ^e_{on}	τ^e_{off}
1	40	0.95	0.05
2	30	0.50	0.50
3	20	0.95	0.05
4	20	0.50	0.50
5	5	0.95	0.05
6	10	0.95	0.05
7	10	0.95	0.05

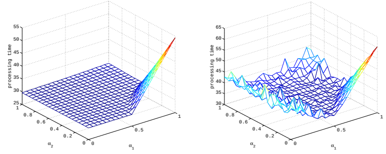
Network graph:



Inflow profil:



Simulation results:



Total production time of 250 units (det. and stochastic simulation) using different sets of distribution rates α_1 and α_2.

References

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This work was supported by BRITISH COUNCIL DAAD Deutscher Akademischer Austausch Dienst German Academic Exchange Service