Asymptotic MS-Stability Analysis for Linear Systems of SDEs

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MS-stability of linear systems

Consider the linear d-dimensional system of SDEs with m multiplicative noise terms

$$\mathrm{d}X(t) = FX(t)\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\mathrm{d}W_r(t), \quad t \geq t_0 \geq 0, \quad X(t_0) = X_0. \qquad (1)$$

Here, $F \in \mathbb{R}^{d\times d}$, $G_1,\dots,G_m \in \mathbb{R}^{d\times d}$, $W = (W_1,\dots,W_m)^T$ is an m-dim. Wiener process. Moreover, Eq. (1) has the *zero solution* $X(t;t_0,0) \equiv 0$ as its *equilibrium solution*.

Definition of asymptotic mean-square stability:

The zero solution of the SDE system (1) is said to be

- mean-square stable, if for each $\varepsilon > 0$, there exists a $\delta \geq 0$ s.t. when $\|X_0\|^2_{L_2} < \delta$,

$$\|X(t;t_0,X_0)\|^2_{L_2} < \varepsilon, \quad t \geq t_0;$$

- asymptotically mean-square stable, if it is MS-stable and, when $\|X_0\|^2_{L_2} < \delta$,

$$\|X(t;t_0,X_0)\|^2_{L_2} \to 0 \quad \text{for} \quad t \to \infty.$$

The second moment of the solution of (1), that is the expectation of the matrix-valued process $P(t) = X(t)X(t)^T$, is given by the matrix ODE

$$\mathrm{dE}(P(t)) = \Big(F\,\mathrm{E}(P(t)) + \mathrm{E}(P(t))F^T + \sum_{r=1}^{m} G_r \mathrm{E}(P(t))G_r^T\Big)\,\mathrm{d}t\,, \quad t \geq t_0 \geq 0\,, \quad P(t_0) = X_0 X_0^T.$$

Applying the vectorisation yields a vector ODE for the d^2-dimensional vector $\mathrm{E}(Y(t)) = \mathrm{E}(\mathrm{vec}(P(t)))$

$$\mathrm{dE}(Y(t)) = S \cdot \mathrm{E}(Y(t))\,\mathrm{d}t\,,$$

where S is the mean-square stability matrix ($\otimes$ denotes the Kronecker product)

$$S = (\mathrm{Id} \otimes F) + (F \otimes \mathrm{Id}) + \sum_{r=1}^{m} (G_r \otimes G_r)\,.$$

Stability condition:

The zero solution of the linear SDE system (1) is asymptotically mean-square stable if and only if

$$\alpha(S) < 0\,.$$

Here, $\alpha(A)$ is the spectral abscissa defined by $\alpha(S) = \max_i \Re(\lambda_i)$, where λ_i are the eigenvalues of the matrix S.

MS-stability of one-step approximations

Consider θ-Maruyama and θ-Milstein approximations given by

$$X_{i+1} = \mathfrak{A}_i X_i, \qquad i = 0,1,\dots\,. \qquad (2)$$

where

$$\mathfrak{A}_i^{\text{Mar}} = A + \sum_{r=1}^{m} B_r \xi_{r,i} \quad \text{and} \quad \mathfrak{A}_i^{\text{Mil}} = \bar{A} + \sum_{r=1}^{m} B_r\,\xi_{r,i} + \sum_{r_1,r_2=1}^{m} C_{r_1,r_2}\,\xi_{r_1,i}\,\xi_{r_2,i}\,,$$

respectively. Here, $\xi_{r,i} \sim \mathcal{N}(0,1)$, $A = (\mathrm{Id} - h\theta F)^{-1}(\mathrm{Id} + h(1-\theta)F)$, $B_r = (\mathrm{Id} - h\theta F)^{-1}\sqrt{h}G_r$, $\bar{A} = A - \sum_{r=1}^{m} C_{r,r}$, $C_{r_1,r_2} = (\mathrm{Id} - h\theta F)^{-1}(0.5h\,G_{r_1}G_{r_2})$, h constant step-size, θ controls the implicity of the drift discretisation and commutative noise.

The second moment of the approximation of (2), that is the expectation of the discrete matrix-valued process $X_i X_i^T$, can be obtained by multiplying (2) with X_{i+1}^T and $(\mathfrak{A}_i X_i)^T$, respectively. Applying the vectorisation and taking expectation yields

$$\mathrm{E}(Y_{i+1}) = \mathcal{S} \cdot \mathrm{E}(Y_i)\,,$$

where $Y_i = \mathrm{vec}(X_i X_i^T)$, and $\mathcal{S} = \mathrm{E}(\mathfrak{A}_i \otimes \mathfrak{A}_i)$ is the mean-square stability matrix of the numerical method

$$\mathcal{S}^{\text{Mar}} = (A \otimes A) + \sum_{r=1}^{m} (B_r \otimes B_r)\,,$$

$$\mathcal{S}^{\text{Mil}} = (A \otimes A) + \sum_{r=1}^{m} (B_r \otimes B_r) + 2\sum_{r=1}^{m} (C_{r,r} \otimes C_{r,r}) + \Bigg(\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} C_{r_1,r_2} \otimes \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{m} C_{r_1,r_2}\Bigg)\,.$$

Stability condition:

The zero solution of the linear system of stochastic difference equations (2) is asymptotically stable in mean-square if and only if

$$\rho(\mathcal{S}) < 1\,.$$

Here, $\rho(\mathcal{S})$ is the spectral radius defined by $\rho(\mathcal{S}) = \max_i |\lambda_i|$, where λ_i are the eigenvalues of the matrix $\mathcal{S}$.

Application to a test-system

Consider the two-dim. test-system with two commutative noise terms:

$$\mathrm{d}X(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X(t)\mathrm{d}t + \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} X(t)\mathrm{d}W_1(t) + \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} X(t)\mathrm{d}W_2(t)\,.$$

Stability conditions: The zero solution of

- this test system is asymptotically mean-square stable if and only if

$$\lambda + \frac{\sigma^2}{2} + \frac{\varepsilon^2}{2} < 0\,;$$

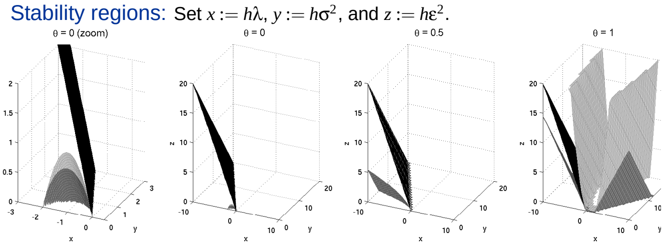
- the stochastic difference equation generated by the θ-Maruyama method is asymptotically mean-square stable if and only if

$$\frac{(1 + (1-\theta)h\lambda)^2 + h(\sigma^2 + \varepsilon^2)}{(1 - \lambda h\theta)^2} < 1\,;$$

- the stochastic difference equation generated by the θ-Milstein method is asymptotically mean-square stable if and only if

$$\frac{(1 + (1-\theta)h\lambda)^2 + h(\sigma^2 + \varepsilon^2) + \frac{1}{2}h^2(\sigma^2 + \varepsilon^2)^2}{(1 - \lambda h\theta)^2} < 1\,.$$

Stability regions: Set $x := h\lambda$, $y := h\sigma^2$, and $z := h\varepsilon^2$.



Boundaries of the mean-square stability regions for the test system (black area) and the θ-Maruyama (light grey area) and the θ-Milstein method (dark grey area) for $\theta = 0, 0.5, 1$. The first plot provides a zoom into the second plot.

Numerical experiments

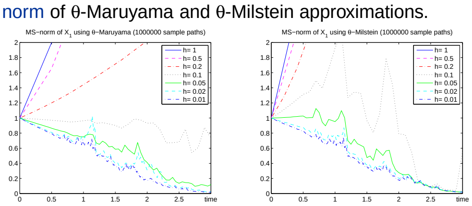
Consider the two-dim. test-system with $\lambda = -2$, $\sigma = 0.5$, $\varepsilon = \sqrt{3}$. The zero-solution of that system is stable with spectral abscissa -0.750.

step-size	θ-Maruyama approximation $\theta = 0$	θ-Maruyama approximation $\theta = 0.5$	θ-Maruyama approximation $\theta = 1$	θ-Milstein approximation $\theta = 0$	θ-Milstein approximation $\theta = 0.5$	θ-Milstein approximation $\theta = 1$
h=1.00	4.250 unst.	0.813 stable	0.472 stable	9.531 unst.	2.133 unst.	1.059 unst.
h=0.50	1.625 unst.	0.833 stable	0.656 stable	2.945 unst.	1.420 unst.	0.986 stable
h=0.20	1.010 unst.	0.896 stable	0.842 stable	1.221 unst.	1.043 unst.	0.950 stable
h=0.10	0.965 stable	0.938 stable	0.920 stable	1.018 unst.	0.982 stable	0.957 stable
h=0.05	0.973 stable	0.966 stable	0.961 stable	0.986 stable	0.978 stable	0.972 stable
h=0.02	0.987 stable	0.986 stable	0.985 stable	0.989 stable	0.988 stable	0.987 stable
h=0.01	0.993 stable	0.993 stable	0.992 stable	0.993 stable	0.993 stable	0.993 stable

Values of the spectral radius of the corresponding stability matrix $\mathcal{S}^{\text{Mar}}$ and $\mathcal{S}^{\text{Mil}}$ for the test-equation computed numerically.

Estimated mean-square norm of θ-Maruyama and θ-Milstein approximations.

Results for the first component X_1 using one million sample paths, $\theta = 0$ and different values of the step-sizes.



References

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